Affine isoperimetric inequalities in the functional Orlicz-Brunn-Minkowski theory *

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Abstract

In this paper, we develop a basic theory of Orlicz affine and geominimal surface areas for convex and s-concave functions. We prove some basic properties for these newly introduced functional affine invariants and establish related functional affine isoperimetric inequalities as well as functional Santaló type inequalities.

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1 Introduction

The definition of Orlicz addition by Gardner, Hug and Weil [15] and Xi, Jin and Leng [33] brings new impulses to the rapidly developing Orlicz-Brunn-Minkowski theory for convex bodies. In fact, Orlicz addition makes it possible to establish the Orlicz-Brunn-Minkowski inequality, develop Orlicz mixed volume, and prove the Orlicz-Minkowski inequality for the Orlicz mixed volume. However, the first steps in this theory were actually the Orlicz affine isoperimetric inequalities for Orlicz centroid bodies and Orlicz projection bodies by Lutwak, Yang, and Zhang [23, 24]. An affine isoperimetric inequality in the Orlicz-Brunn-Minkowski theory provides upper and/or lower bounds, in terms of volume, for functionals defined on convex bodies which are invariant under all volume preserving linear transforms; and it would be ideal if these functionals attain their maximum or minimum at (and only at) ellipsoids. It is convenient and natural to call affine isoperimetric inequalities in the Orlicz-Brunn-Minkowski theory as Orlicz affine isoperimetric inequalities, just like the L_p affine isoperimetric inequalities in the L_p-Brunn-Minkowski theory. Another example of Orlicz affine isoperimetric inequalities is the one by the second author [36], which provides bounds for Orlicz affine and geominimal surface areas, that is, under certain conditions, Orlicz affine and geominimal surface areas attain their maximum (or minimum) at and only at ellipsoids.

Developing and extending affine surface areas has been a central goal in convex geometry for decades. The following are the major steps. The first major step was due to Blaschke [6], who defined the classical L_1 affine surface area. Then, Lutwak [22] introduced L_p affine surface areas for $p>1$. Based on some beautiful integral formulas for L_p affine surface areas (which essentially involve Gauss curvature and

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the support function), Schütt and Werner [31] proposed a further extension of L_p affine surface area to $-n \neq p \in \mathbb{R}$. Later, Ludwig and Reitzner [21] and Ludwig [20] introduced the general affine surface areas for non-homogeneous functions. Note that the above affine surface areas are not continuous with respect to the Hausdorff metric. However, the classical L_1 geominimal surface area, which is closely related to the classical L_1 affine surface area, was proved to be continuous with respect to the Hausdorff metric and to be a bridge between several different type of geometries (see Petty [27] for more details). Since there are no convenient integral formulas for L_p geominimal surface areas for $p > 1$, for the definition of the L_p geominimal surface area for $-n \neq p \in \mathbb{R}$, a different approach from those used in [20, 21, 31] is needed; and that was proposed in [37] (actually, such an approach was motivated by Lutwak's definition of the L_p geominimal surface area for $p > 1$ [22] and the work [34] by Xiao). In fact, the approach in [37] also provides alternative definitions for the L_p affine surface areas for $-n \neq p \in \mathbb{R}$. This opens the door to develop Orlicz affine and geominimal surface areas [36], as well as their duals for star bodies [38] (based on the dual Orlicz mixed volume in [16]) and their mixed counterparts involving multiple convex bodies [36, 39]. See e.g., [20, 22, 28, 32, 35, 36, 37] for affine isoperimetric inequalities related to affine and geominimal surface areas.

The geometry of log-concave functions aims to study the geometric properties of log-concave functions, in a manner similar to the geometry of convex bodies (also known as convex geometry or the Brunn-Minkowski theory of convex bodies). In fact, there is a "dictionary" between these two theories, for instance, integral translates to volume, log-concave functions to convex bodies, the Gaussian function $e^{-\frac{\|\cdot\|^2}{2}}$ to the unit Euclidean ball, polar duals of log-concave functions to polars of convex bodies, and the integral product to the Mahler volume product. The geometry of log-concave functions extends fundamental notions and results in convex geometry nontrivially to their functional counterparts. Moreover, it usually provides much more powerful tools and far-reaching results than its geometric counterpart (indeed, every convex body can be associated with a log-concave function). See, e.g., Klartag and Milman [17] and Milman [26] for more detailed motivation and references.

An important functional affine isoperimetric inequality is the functional Blaschke-Santaló inequality [2, 5, 13, 18, 19], which is essential for the isoperimetric inequalities for L_p affine surface areas of log-concave and s-concave functions [10, 11, 12]. In their seminal paper [3], Artstein-Avidan, Klartag, Schütt and Werner provided a definition of L_1 affine surface area for s-concave functions and established related functional affine isoperimetric inequality. In particular, a functional affine isoperimetric inequality for log-concave functions was given and can be viewed as an inverse logarithmic Sobolev inequality for entropy. These inequalities further imply a version of the reverse Poincaré inequality [3]. The main purpose of this paper is to develop a theory of Orlicz affine and geominimal surface areas for convex functions (hence also for log-concave functions) as well as their related functional affine isoperimetric inequalities. The results in this paper bring more items into the above mentioned "dictionary" and hopefully will provide powerful tools for many related fields, such as, analysis, (convex) geometry, and information theory.

This paper is organized as follows. In Section 2, we give a new formula for a general functional L_p affine surface area for convex functions. Then, we generalize this idea and introduce the general Orlicz affine and geominimal surface areas for convex functions. We prove that these new concepts are $SL_{\pm}(n)$-invariant. We also prove some inequalities for these notions, such as functional affine isoperimetric inequalities, and generalizations of functional Blaschke-Santaló and inverse Santaló inequalities. In Section 3, we propose the definition of Orlicz affine and geominimal surface areas for s-concave functions

and prove corresponding functional inequalities, e.g., functional affine isoperimetric and Santaló type inequalities. In Section 4, we will briefly discuss results for multiple convex functions.

2 The general Orlicz affine and geominimal surface areas for convex functions

Let $(\mathbb{R}^n, \|\cdot\|)$ be the Euclidean space with $\|\cdot\|$ the Euclidean norm of $\mathbb{R}^n$ induced by the usual inner product $\langle\cdot,\cdot\rangle$. Let $\mathcal{C}$ be the set of all convex functions $\psi:\mathbb{R}^n\to\mathbb{R}\cup\{+\infty\}$. Throughout this paper, the interior of the convex domain of $\psi\in\mathcal{C}$ is always assumed to be nonempty. Denote by ψ^* the classical Legendre transform of ψ, that is,

$$\psi^*(y)=\sup_{x\in\mathbb{R}^n}\big(\langle x,y\rangle-\psi(x)\big). \tag{1}$$

Clearly, $\psi(x)+\psi^*(y)\geq\langle x,y\rangle$ for all $x,y\in\mathbb{R}^n$. Equality holds if and only if x is in the domain of ψ and y is in the subdifferential of ψ at x: for almost all x in the domain of $\psi\in\mathcal{C}$,

$$\psi^*(\nabla\psi(x))=\langle x,\nabla\psi(x)\rangle-\psi(x),$$

where $\nabla\psi$ denotes the gradient of ψ. Rademacher's theorem (e.g., [8]) asserts that $\nabla\psi$ exists almost everywhere. For $\psi\in\mathcal{C}$, $\nabla^2\psi$ denotes the Hessian matrix of ψ in the sense of Alexandrov, and it exists almost everywhere by a theorem of Alexandrov [1] and Busemann-Feller [9]. Let

$$X_\psi=\Big\{x\in\mathbb{R}^n:\ \psi(x)<\infty,\ \text{and } \nabla^2\psi(x) \text{ exists and is invertible}\Big\}.$$

For more background on convex functions, please see [25, 29, 30].

Denote by f° the polar dual of the function $f:\mathbb{R}^n\to[0,\infty)$, which has the form:

$$f^\circ(x)=\inf_{y\in\mathbb{R}^n}\Big(\frac{e^{-\langle x,y\rangle}}{f(y)}\Big)\quad\Leftrightarrow\quad -\log f^\circ=(-\log f)^*.$$

A function $f:\mathbb{R}^n\to[0,\infty)$ is log-concave if $\log f$ is concave on the support of f. Note that f° is always a log-concave function no matter whether f is log-concave or not. A log-concave function f is often written as $f=e^{-\psi}$ with $\psi\in\mathcal{C}$, and clearly $f^\circ=e^{-\psi^*}$. Moreover, $(f^\circ)^\circ=f$ if f is an upper semi-continuous log-concave function. The function $\gamma_n=e^{-\frac{\|\cdot\|^2}{2}}$ serves as the "unit Euclidean ball" of log-concave functions as $(\gamma_n)^\circ=\gamma_n$, and its integral over $\mathbb{R}^n$ is equal to $(\sqrt{2\pi})^n$.

Throughout the paper, we always assume that the functions we consider, such as $F_1,F_2:\mathbb{R}\to(0,\infty)$ and $\psi\in\mathcal{C}$, have enough smoothness and integrability to guarantee the integrals or other expression well-defined. For instance, we will need the following integrals to be finite

$$0<\int_{X_\psi}F_1(\psi(x))\,dx<\infty\quad\text{and}\quad 0<\int_{X_{\psi^*}}F_2(\psi^*(y))dy<\infty.$$

2.1 A new formula for a general L_p affine surface area for convex functions

The following general L_p affine surface area for convex functions was proposed in [10].

Definition 1. *For measurable functions $F_1, F_2 \colon \mathbb{R} \to (0,\infty)$, $-n \neq p \in \mathbb{R}$, and $\psi \in \mathcal{C}$, define*

$$as_{p,F_1,F_2}(\psi) = \int_{X_\psi} \big(F_1(\psi(x))\big)^{\frac{n}{n+p}} \big(F_2(\langle x, \nabla\psi(x)\rangle - \psi(x))\, \det \nabla^2\psi(x)\big)^{\frac{p}{n+p}}\, dx. \tag{2}$$

Remark. Note that $as_{p,F_1,F_2}(\cdot)$ is called the general L_p affine surface area because the above definition is just the definition of the functional L_p affine surface area for log-concave functions if $F_1(t) = F_2(t) = e^{-t}$. Hence, functions F_1 and F_2 act like parameters and provide the power to include much wider class of functions than the log-concave functions.

Denote by $\mathcal{F}^+_{\psi^*}$ the set of all positive Lebesgue integrable functions defined on X_{ψ^*}. That is, $g \in \mathcal{F}^+_{\psi^*}$ if $g(y) > 0$ for all $y \in X_{\psi^*}$ and $0 < I(g, \psi^*) < \infty$ with

$$I(g,\psi^*) = \int_{X_{\psi^*}} g(y)\, dy = \int_{X_\psi} g(\nabla\psi(x)) \det \nabla^2\psi(x)\, dx, \tag{3}$$

where the second equality follows from Corollary 4.3 and Proposition A.1 in [25]. In particular,

$$\begin{aligned} I(F_2 \circ \psi^*, \psi^*) &= \int_{X_{\psi^*}} F_2(\psi^*(y)) dy \\ &= \int_{X_\psi} F_2(\psi^*(\nabla\psi(x))) \det \nabla^2\psi(x) dx \\ &= \int_{X_\psi} F_2(\langle x, \nabla\psi(x)\rangle - \psi(x)) \det \nabla^2\psi(x) dx. \end{aligned}$$

We often need $I(F_1 \circ \psi, \psi) = \int_{X_\psi} F_1(\psi(x))\, dx$.

For measurable functions $F_1, F_2 : \mathbb{R} \to (0,\infty)$, let

$$V_{p,F_1,F_2}(\psi, g) = \int_{X_\psi} \left(\frac{F_2(\langle x, \nabla\psi(x)\rangle - \psi(x))}{g(\nabla\psi(x))} \right)^{p/n} F_1(\psi(x))\, dx. \tag{4}$$

The following theorem gives a new formula for the above general functional L_p affine surface area.

Theorem 1. *Let ψ be a C^2 strictly convex function. For $p \geq 0$, one has*

$$as_{p,F_1,F_2}(\psi) = \inf_{g \in \mathcal{F}^+_{\psi^*}} \left\{ V_{p,F_1,F_2}(\psi, g)^{\frac{n}{n+p}}\ I(g,\psi^*)^{\frac{p}{n+p}} \right\},$$

while for $-n \neq p < 0$, the above formula holds with "$\inf$" replaced by "$\sup$".

Proof. We only prove the desired result for $p \in (0, \infty)$. The result for $-n \neq p < 0$ follows along the same lines and for $p = 0$ holds trivially.

As ψ is a C^2 strictly convex function, then $\det\nabla^2\psi(x) > 0$ on X_ψ and $\nabla\psi : X_\psi \to X_{\psi^*}$ is smooth and bijective. Consider the following function

$$g_0(\nabla\psi(x)) = \big[F_2(\langle x, \nabla\psi(x)\rangle - \psi(x))\big]^{\frac{p}{n+p}} \left(\frac{F_1(\psi(x))}{\det\nabla^2\psi(x)}\right)^{\frac{n}{n+p}} \qquad \text{for } x \in X_\psi.$$

By formulas (2), (3) and (4), one can check

$$as_{p,F_1,F_2}(\psi) = \big[V_{p,F_1,F_2}(\psi, g_0)\big]^{\frac{n}{n+p}} I(g_0, \psi^*)^{\frac{p}{n+p}} \ge \inf_{g\in\mathcal{F}^+_{\psi^*}} [V_{p,F_1,F_2}(\psi, g)]^{\frac{n}{n+p}} I(g, \psi^*)^{\frac{p}{n+p}}.$$

On the other hand, Hölder's inequality implies that for all $g \in \mathcal{F}^+_{\psi^*}$,

$$\begin{aligned} as_{p,F_1,F_2}(\psi) &= \int_{X_\psi}\left[F_1(\psi(x))\left(\frac{F_2(\langle x, \nabla\psi(x)\rangle - \psi(x))}{g(\nabla\psi(x))}\right)^{\frac{p}{n}}\right]^{\frac{n}{n+p}} \big(g(\nabla\psi(x))\det\nabla^2\psi(x)\big)^{\frac{p}{n+p}} dx \\ &\le \big[V_{p,F_1,F_2}(\psi, g)\big]^{\frac{n}{n+p}} I(g, \psi^*)^{\frac{p}{n+p}}. \end{aligned} \tag{5}$$

Taking the infimum over all $g \in \mathcal{F}^+_{\psi^*}$, one gets, for $p \in (0, \infty)$,

$$as_{p,F_1,F_2}(\psi) \le \inf_{g\in\mathcal{F}^+_{\psi^*}} [V_{p,F_1,F_2}(\psi, g)]^{\frac{n}{n+p}} I(g, \psi^*)^{\frac{p}{n+p}},$$

and hence the desired result holds. □

Remark. Let $y = \nabla\psi(x)$, then $\psi(x) + \psi^*(y) = \langle x, y\rangle$, $x = \nabla\psi^*(y)$ and $\nabla^2\psi(x)\nabla^2\psi^*(y) = \mathrm{Id}$ (the identity matrix on $\mathbb{R}^n$). These lead to the explicit expression of g_0:

$$g_0(y) = \big[F_1(\langle y, \nabla\psi^*(y)\rangle - \psi^*(y)) \cdot F_2(\psi^*(y))^{\frac{p}{n}} \cdot \det\nabla^2\psi^*(y)\big]^{\frac{n}{n+p}}.$$

2.2 The general Orlicz affine and geominimal surface areas for convex functions

Let $h : (0, \infty) \to (0, \infty)$ be a continuous function and $\psi \in \mathcal{C}$.

Definition 2. *For measurable functions $F_1, F_2 : \mathbb{R} \to (0, \infty)$ and $g \in \mathcal{F}^+_{\psi^*}$, define the Orlicz mixed integral of ψ and g with respect to F_1 and F_2 by*

$$V_{h,F_1,F_2}(\psi, g) = \int_{X_\psi} h\left(\frac{g(\nabla\psi(x))}{F_2(\langle x, \nabla\psi(x)\rangle - \psi(x))}\right) F_1(\psi(x))\, dx.$$

When $h(t) = t^{-p/n}$, one recovers formula (4). Moreover, if $g = \tau\cdot(F_2 \circ \psi^*)$ for some constant $\tau > 0$,

$$V_{h,F_1,F_2}\big(\psi, \tau\cdot(F_2 \circ \psi^*)\big) = h(\tau)\cdot I(F_1 \circ \psi, \psi). \tag{6}$$

Denote by $GL(n)$ the set of all invertible linear maps on $\mathbb{R}^n$. For $T \in GL(n)$, we use $\det(T)$ or $\det T$ for the determinant of T. Let $SL_\pm(n)$ denote the subset of $GL(n)$ which contains all $T \in GL(n)$ such

that $\det(T) = \pm 1$. The inverse of T is written by T^{-1} and the transpose of T is written as T^t. For convenience, the inverse of T^t is denoted by T^{-t}.

For $T \in SL_{\pm}(n)$ and $g \in \mathcal{F}^+_{\psi^*}$, by formula (1), one has,

$$\begin{aligned} (\psi \circ T)^*(y) &= \sup_{x \in \mathbb{R}^n} \big(\langle x, y\rangle - (\psi \circ T)(x)\big) = \sup_{x \in \mathbb{R}^n} \big(\langle Tx, T^{-t}y\rangle - \psi(Tx)\big) \\ &= \psi^*(T^{-t}y) = \big(\psi^* \circ T^{-t}\big)(y). \end{aligned}$$

Hence, $y \in X_{(\psi\circ T)^*}$ if and only if $y \in T^t(X_{\psi^*})$, which follows from the general fact

$$\nabla^2(\psi \circ T)(x) = T^t\big(\nabla^2\psi(Tx)\big)T$$

for $T \in GL(n)$. This implies $g \circ T^{-t} \in \mathcal{F}^+_{(\psi\circ T)^*}$ if $g \in \mathcal{F}^+_{\psi^*}$. Moreover, by $|\det(T^{-t})| = 1$ and by $y = T^{-t}z$, formula (3) implies

$$\begin{aligned} I(g, \psi^*) &= \int_{X_{\psi^*}} g(y)\, dy = \int_{T^t(X_{\psi^*})} g(T^{-t}z)\, dz \\ &= \int_{X_{(\psi\circ T)^*}} (g \circ T^{-t})(z)\, dz = I\big(g \circ T^{-t}, (\psi \circ T)^*\big). \end{aligned} \tag{7}$$

Moreover, we can prove that the above defined Orlicz mixed integral is $SL_{\pm}(n)$-invariant.

Lemma 1. *Let F_1, F_2, ψ and g be as in Definition 2. Then, for all $T \in SL_{\pm}(n)$, one has,*

$$V_{h,F_1,F_2}(\psi \circ T, g \circ T^{-t}) = V_{h,F_1,F_2}(\psi, g).$$

Proof. Let $T \in SL_{\pm}(n)$. Recall that $\nabla(\psi \circ T)(x) = T^t\nabla\psi(Tx)$, which implies $x \in X_{\psi\circ T}$ if and only if $Tx \in X_\psi$. Hence, by letting $y = Tx$, one has,

$$\begin{aligned} V_{h,F_1,F_2}(\psi \circ T, g \circ T^{-t}) &= \int_{X_{\psi\circ T}} h\bigg(\frac{(g \circ T^{-t})\big(\nabla(\psi \circ T)(x)\big)}{F_2(\langle x, \nabla(\psi \circ T)(x)\rangle - (\psi \circ T)(x))}\bigg) F_1\big((\psi \circ T)(x)\big)\, dx \\ &= \int_{X_{\psi\circ T}} h\left(\frac{g(\nabla\psi(Tx))}{F_2(\langle x, T^t\nabla\psi(Tx)\rangle - \psi(Tx))}\right) F_1(\psi(Tx))\, dx \\ &= \int_{X_\psi} h\left(\frac{g(\nabla\psi(y))}{F_2(\langle y, \nabla\psi(y)\rangle - \psi(y))}\right) F_1(\psi(y))\, dy \\ &= V_{h,F_1,F_2}(\psi, g). \end{aligned}$$

□

The following function classes were defined in [36] and will play fundamental roles in this paper. Let

$$\begin{aligned} \Phi &= \{h : h \text{ is either a constant or a strictly convex function}\}; \\ \Psi &= \{h : h \text{ is either a constant or an increasing strictly concave function}\}. \end{aligned}$$

Throughout this paper, $\mathcal{L}_{\psi^*}$ refers to the subset of $\mathcal{F}^+_{\psi^*}$ which contains all log-concave functions. Note that log-concave functions are analogous to convex bodies in geometry; and hence $\mathcal{L}_{\psi^*}$ is used to define the general Orlicz geominimal surface area of convex functions (although ψ or $F_1 \circ \psi$ or $F_2 \circ \psi^*$ may not be log-concave). Motivated by Theorem 1, the general Orlicz affine and geominimal surface areas of ψ could be defined as follows.

Definition 3. *For $h \in \Phi$, the general Orlicz affine surface area of $\psi \in \mathcal{C}$ is defined by*

$$as_{h,F_1,F_2}^{orlicz}(\psi) = \inf_{g \in \mathcal{F}_{\psi^*}^+} \left\{ V_{h,F_1,F_2}\left(\psi, \frac{(\sqrt{2\pi})^n \cdot g}{I(g,\psi^*)}\right) \right\};$$

and the general Orlicz geominimal surface area of $\psi \in \mathcal{C}$ is defined by

$$G_{h,F_1,F_2}^{orlicz}(\psi) = \inf_{g \in \mathcal{L}_{\psi^*}} \left\{ V_{h,F_1,F_2}\left(\psi, \frac{(\sqrt{2\pi})^n \cdot g}{I(g,\psi^*)}\right) \right\}.$$

When $h \in \Psi$, $as_{h,F_1,F_2}^{orlicz}(\psi)$ and $G_{h,F_1,F_2}^{orlicz}(\psi)$ are defined as above but with "$\inf$" replaced by "$\sup$".

Remark. The above definitions could be extended to more general cases with $\mathcal{F}_{\psi^*}^+$ and $\mathcal{L}_{\psi^*}$ replaced by any subset of $\mathcal{F}_{\psi^*}^+$; and the properties would be similar to those for $as_{h,F_1,F_2}^{orlicz}(\psi)$ and $G_{h,F_1,F_2}^{orlicz}(\psi)$ which are the most important cases. In fact, one can let $I(g,\psi^*) = (\sqrt{2\pi})^n$ in Definition 3, for instance, if $h \in \Phi$,

$$as_{h,F_1,F_2}^{orlicz}(\psi) = \inf\left\{V_{h,F_1,F_2}(\psi, g) : g \in \mathcal{F}_{\psi^*}^+ \text{ with } I(g,\psi^*) = (\sqrt{2\pi})^n\right\}.$$

It can be also easily checked that $as_{h,F_1,F_2}^{orlicz}(\psi) \leq G_{h,F_1,F_2}^{orlicz}(\psi)$ for $h \in \Phi$ and $as_{h,F_1,F_2}^{orlicz}(\psi) \geq G_{h,F_1,F_2}^{orlicz}(\psi)$ for $h \in \Psi$. Moreover, $as_{h,F_1,F_2}^{orlicz}(\psi) = G_{h,F_1,F_2}^{orlicz}(\psi) = I(F_1 \circ \psi, \psi)$ if $h(t) = 1$.

If $F_1(t) = F_2(t) = e^{-t}$ and ψ is a convex function, then $f = F_1 \circ \psi = e^{-\psi}$ and $F_2 \circ \psi^* = e^{-\psi^*} = f^\circ$ (the polar dual of f) are log-concave functions. Therefore, one can define the Orlicz affine surface area of the log-concave function $f = e^{-\psi}$ by $as_h^{orlicz}(f) = as_{h,e^{-t},e^{-t}}^{orlicz}(\psi)$. This serves as a non-homogeneous extension of the L_p affine surface area of log-concave functions [10, 11]. Similarly, $G_h^{orlicz}(f) = G_{h,e^{-t},e^{-t}}^{orlicz}(\psi)$ defines the Orlicz geominimal surface area of f, which is new to the literature.

The following theorem states that the general Orlicz affine and geominimal surface areas of ψ are $SL_{\pm}(n)$-invariant.

Theorem 2. *Let $\psi \in \mathcal{C}$. For $T \in SL_{\pm}(n)$ and $h \in \Phi \cup \Psi$, one has*

$$as_{h,F_1,F_2}^{orlicz}(\psi \circ T) = as_{h,F_1,F_2}^{orlicz}(\psi) \quad \text{and} \quad G_{h,F_1,F_2}^{orlicz}(\psi \circ T) = G_{h,F_1,F_2}^{orlicz}(\psi).$$

In particular, $as_h^{orlicz}(f)$ and $G_h^{orlicz}(f)$ are $SL_{\pm}(n)$-invariant.

Proof. We only prove the case for $as_{h,F_1,F_2}^{orlicz}(\psi)$ and the case for $G_{h,F_1,F_2}^{orlicz}(\psi)$ follows along the same lines. The desired result follows from Lemma 1, formula (7) and the remark after Definition 3: for $h \in \Phi$,

$$\begin{aligned}
as_{h,F_1,F_2}^{orlicz}(\psi) &= \inf\left\{V_{h,F_1,F_2}(\psi, g) : g \in \mathcal{F}_{\psi^*}^+ \text{ with } I(g,\psi^*) = (\sqrt{2\pi})^n\right\} \\
&= \inf\left\{V_{h,F_1,F_2}(\psi \circ T, g \circ T^{-t}) : g \in \mathcal{F}_{\psi^*}^+ \text{ with } I(g,\psi^*) = (\sqrt{2\pi})^n\right\} \\
&= \inf\left\{V_{h,F_1,F_2}(\psi \circ T, g \circ T^{-t}) : g \circ T^{-t} \in \mathcal{F}_{(\psi\circ T)^*}^+ \text{ with } I\big(g \circ T^{-t}, (\psi \circ T)^*\big) = (\sqrt{2\pi})^n\right\} \\
&= as_{h,F_1,F_2}^{orlicz}\big(\psi \circ T\big).
\end{aligned}$$

Replacing "inf" by "sup", one gets the $SL_{\pm}(n)$-invariance of $as_{h,F_1,F_2}^{orlicz}(\psi)$ for $h \in \Psi$. $\square$

Let $c>0$ be a constant and $F:\mathbb{R}\to(0,\infty)$ be a measurable function. For convenience, let

$$I(F,c)=\int_{\mathbb{R}^n} F\Big(\frac{c^2\|x\|^2}{2}\Big)\,dx.$$

It can be checked that

$$I(F,c)=c^{-n}\cdot I(F,1). \tag{8}$$

The following corollary provides the precise values of $as^{orlicz}_{h,aF,bF}\big(\frac{c^2\|\cdot\|^2}{2}\big)$ and $G^{orlicz}_{h,aF,bF}\big(\frac{c^2\|\cdot\|^2}{2}\big)$ with constants $a,b>0$. When $a=b=1$ and $F(t)=e^{-t}$, one gets

$$as^{orlicz}_{h}\big(\gamma_n\circ c\big)=G^{orlicz}_{h}\big(\gamma_n\circ c\big)=c^{-n}\cdot h(c^{-n})\cdot(\sqrt{2\pi})^n,$$

where $(\gamma_n\circ c)(x)=\gamma_n(cx)$ for $x\in\mathbb{R}^n$ and $\gamma_n(x)=e^{-\frac{\|x\|^2}{2}}$. Note that $\gamma_n^\circ=\gamma_n$, and hence γ_n serves as the "Euclidean unit ball" in the geometry of log-concave functions.

Corollary 1. *Let $a,b,c>0$ be constants and $F\colon\mathbb{R}\to(0,\infty)$ be a measurable function such that $0<I\big(F,1\big)<\infty$. Then, for $h\in\Phi\cup\Psi$,*

$$as^{orlicz}_{h,aF,bF}\Big(\frac{c^2\|\cdot\|^2}{2}\Big)=a\cdot I\big(F,c\big)\cdot h\Big(\frac{(\sqrt{2\pi})^n}{c^{2n}\cdot b\cdot I(F,c)}\Big).$$

The same formula holds for $G^{orlicz}_{h,aF,bF}\big(\frac{c^2\|\cdot\|^2}{2}\big)$ if the function $F\big(\frac{\|\cdot\|^2}{2}\big)$ is log-concave.

Proof. We only prove the case $h\in\Phi$ and the proof for the case $h\in\Psi$ follows along the same line. Note that $X_{\frac{c^2\|\cdot\|^2}{2}}=\mathbb{R}^n$, $\nabla\frac{c^2\|x\|^2}{2}=c^2x$ and $\langle x,\nabla\frac{c^2\|x\|^2}{2}\rangle-\frac{c^2\|x\|^2}{2}=\frac{c^2\|x\|^2}{2}$. Applying Jensen's inequality to the convex function h, one has, for all $g:\mathbb{R}^n\to(0,\infty)$ with $\int_{\mathbb{R}^n}g(y)\,dy>0$,

$$\begin{aligned}
V_{h,aF,bF}\Big(\frac{c^2\|\cdot\|^2}{2},g\Big) &= a\cdot I\big(F,c\big)\cdot\int_{\mathbb{R}^n} h\bigg(\frac{g(c^2x)}{b\cdot F\big(\frac{c^2\|x\|^2}{2}\big)}\bigg)\frac{F\big(\frac{c^2\|x\|^2}{2}\big)}{I\big(F,c\big)}\,dx\\
&\geq a\cdot I\big(F,c\big)\cdot h\bigg(\int_{\mathbb{R}^n}\frac{g(c^2x)}{b\cdot I\big(F,c\big)}\,dx\bigg)\\
&= a\cdot I\big(F,c\big)\cdot h\bigg(\frac{1}{c^{2n}\cdot b\cdot I\big(F,c\big)}\int_{\mathbb{R}^n}g(y)\,dy\bigg).
\end{aligned}$$

This leads to, for $h\in\Phi$,

$$\begin{aligned}
as^{orlicz}_{h,aF,bF}\Big(\frac{c^2\|\cdot\|^2}{2}\Big) &= \inf\bigg\{V_{h,aF,bF}\Big(\frac{c^2\|\cdot\|^2}{2},g\Big)\colon g \text{ is a positive function on } \mathbb{R}^n \text{ and } \int_{\mathbb{R}^n}g(y)\,dy=(\sqrt{2\pi})^n\bigg\}\\
&\geq a\cdot I\big(F,c\big)\cdot h\bigg(\frac{(\sqrt{2\pi})^n}{c^{2n}\cdot b\cdot I\big(F,c\big)}\bigg).
\end{aligned}$$

On the other hand, by formulas (6) and (8), and Definitions 2 and 3, one can check

$$as^{orlicz}_{h,aF,bF}\Big(\frac{c^2\|\cdot\|^2}{2}\Big)\leq V_{h,aF,bF}\Big(\frac{c^2\|\cdot\|^2}{2},\frac{(\sqrt{2\pi})^n\cdot F\big(\frac{\|\cdot\|^2}{2c^2}\big)}{I(F,c^{-1})}\Big)=a\cdot I\big(F,c\big)\cdot h\bigg(\frac{(\sqrt{2\pi})^n}{c^{2n}\cdot b\cdot I\big(F,c\big)}\bigg), \tag{9}$$

and the desired result follows.

The proof for $G^{orlicz}_{h,aF,bF}\big(\frac{c^2\|\cdot\|^2}{2}\big)$ follows along the same lines. The additional assumption that $F\big(\frac{\|\cdot\|^2}{2}\big)$ is log-concave is needed to obtain inequality (9). $\square$

2.3 Inequalities

In this subsection, we prove some inequalities for the general Orlicz affine and geominimal surface areas of convex functions. Hereafter, we always assume that

$$I(F_1\circ\psi,\psi)\in(0,\infty)\quad\text{and}\quad I(F_2\circ\psi^*,\psi^*)\in(0,\infty).$$

In particular, when $F_1(t)=F_2(t)=e^{-t}$, we assume that

$$I(f)=I(e^{-t}\circ\psi,\psi)\in(0,\infty)\ \text{ and }\ I(f^\circ)=I(e^{-t}\circ\psi^*,\psi^*)\in(0,\infty),$$

where $f=e^{-\psi}$ and $f^\circ=e^{-\psi^*}$ are log-concave functions.

The following proposition is needed in order to prove some inequalities for the general Orlicz affine and geominimal surface areas of convex functions.

Proposition 1. *Let $\psi\in\mathcal{C}$. Then, for $h\in\Phi$,*

$$as^{orlicz}_{h,F_1,F_2}(\psi)\le I(F_1\circ\psi,\psi)\cdot h\Big(\frac{(\sqrt{2\pi})^n}{I(F_2\circ\psi^*,\psi^*)}\Big),$$

and if in addition $F_2\circ\psi^*$ is log-concave,

$$as^{orlicz}_{h,F_1,F_2}(\psi)\le G^{orlicz}_{h,F_1,F_2}(\psi)\le I(F_1\circ\psi,\psi)\cdot h\Big(\frac{(\sqrt{2\pi})^n}{I(F_2\circ\psi^*,\psi^*)}\Big).$$

In particular, for $h\in\Phi$ and $f=e^{-\psi}$,

$$as^{orlicz}_{h}(f)\le G^{orlicz}_{h}(f)\le I(f)\cdot h\left(\frac{(\sqrt{2\pi})^n}{I(f^\circ)}\right).$$

The above inequalities hold for $h\in\Psi$ with "$\le$" replaced by "$\ge$".

Proof. Formula (6) and Definition 3 imply that for $h\in\Phi$,

$$as^{orlicz}_{h,F_1,F_2}(\psi)\le V_{h,F_1,F_2}\left(\psi,\frac{(\sqrt{2\pi})^n\cdot(F_2\circ\psi^*)}{I(F_2\circ\psi^*,\psi^*)}\right)=I(F_1\circ\psi,\psi)\cdot h\Big(\frac{(\sqrt{2\pi})^n}{I(F_2\circ\psi^*,\psi^*)}\Big);$$

while for $h\in\Psi$, similar inequality holds with "$\le$" replaced by "$\ge$".

The desired result for $G^{orlicz}_{h,F_1,F_2}(\psi)$ follows along the same lines if in addition $F_2\circ\psi^*\in\mathcal{L}_{\psi^*}$. $\square$

For measurable functions $F_1,F_2:\mathbb{R}\to(0,\infty)$, define the decreasing function $\breve{F}:\mathbb{R}\to(0,\infty)$ by

$$\breve{F}(t)=\sup_{\frac{t_1+t_2}{2}\ge t}\sqrt{F_1(t_1)F_2(t_2)}.$$

It can be checked that $\breve{F} = F_1 = F_2$ if $F_1 = F_2$ is a log-concave and decreasing function. Let

$$I(\breve{F}, c) = \int_{\mathbb{R}^n} \breve{F}\Big(\frac{c^2\|x\|^2}{2}\Big)\, dx.$$

For $z \in \mathbb{R}^n$ and for $\psi \in \mathcal{C}$, let $\psi_z(x) = \psi(x+z)$ and $\psi_z^* = (\psi_z)^*$. It was proved in [10] (as a direct consequence of the functional Blaschke-Santaló inequality [13, 19]) that there exists $z_0 \in \mathbb{R}^n$ such that

$$I(F_1 \circ \psi, \psi) \cdot I(F_2 \circ \psi_{z_0}^*, \psi_{z_0}^*) \leq \big[I\big(\breve{F}, 1\big)\big]^2.$$

Let $\mathcal{C}_0$ be the set of convex functions in $\mathcal{C}$ with $z_0 = 0$. Therefore, for all $\psi \in \mathcal{C}_0$, one has

$$I(F_1 \circ \psi, \psi) \cdot I(F_2 \circ \psi^*, \psi^*) \leq \big[I\big(\breve{F}, 1\big)\big]^2. \tag{10}$$

If in addition $\breve{F}$ is strictly decreasing and $I(F_1 \circ \psi, \psi) \neq 0$ (or $I(F_2 \circ \psi^*, \psi^*) \neq 0$), equality holds in inequality (10) if and only if there exist $b \in (0, \infty)$, $a \in \mathbb{R}$ and a positive definite matrix A such that for every $x \in \mathbb{R}^n$ and $t \geq 0$,

$$\psi(x) = \langle Ax, x\rangle + a, \quad F_1(t+a) = b\breve{F}(t) \quad \text{and} \quad bF_2(t-a) = \breve{F}(t). \tag{11}$$

In particular, for log-concave function $f = e^{-\psi}$, inequality (10) becomes the classical functional Blaschke-Santaló inequality [2, 5, 13, 18]:

$$I(f) \cdot I(f^\circ) \leq (2\pi)^n,$$

with equality if and only if there exist $a \in \mathbb{R}$ and a positive definite matrix A such that

$$\psi(x) = \langle Ax, x\rangle + a, \qquad \text{for } x \in \mathbb{R}^n. \tag{12}$$

Now we can prove the following functional affine isoperimetric inequalities, which provide upper bound (lower bound, respectively) for the general Orlicz affine and geominimal surface areas for $h \in \Phi$ (for $h \in \Psi$ respectively). For convenience, let

$$\hat{c} = \Big(\frac{I(\breve{F}, 1)}{I(F_2 \circ \psi^*, \psi^*)}\Big)^{\frac{1}{n}} \text{ and } \bar{c} = \Big(\frac{I(\breve{F}, 1)}{I(F_1 \circ \psi, \psi)}\Big)^{\frac{1}{n}}.$$

Theorem 3. *Let $F_1, F_2 \colon \mathbb{R} \to (0, \infty)$ be measurable functions such that $0 < I(\breve{F}, 1) < \infty$. Let $\psi \in \mathcal{C}_0$.*
(i) For $h \in \Phi$, one has,

$$as^{orlicz}_{h,F_1,F_2}(\psi) \leq as^{orlicz}_{h,\breve{F},\breve{F}}\Big(\frac{\|\cdot\|^2}{2 \cdot \hat{c}^2}\Big),$$

and if in addition both $F_2 \circ \psi^*$ and $\breve{F}(\frac{\|\cdot\|^2}{2})$ are log-concave,

$$G^{orlicz}_{h,F_1,F_2}(\psi) \leq G^{orlicz}_{h,\breve{F},\breve{F}}\Big(\frac{\|\cdot\|^2}{2 \cdot \hat{c}^2}\Big).$$

If in addition $\breve{F}$ is strictly decreasing, equality holds if and only if $F_1, F_2, \breve{F}, \psi$ satisfy formula (11).

(ii) For $h \in \Phi$ being a decreasing function, one has,

$$as^{orlicz}_{h,F_1,F_2}(\psi) \le as^{orlicz}_{h,\breve{F},\breve{F}}\Big(\frac{\bar{c}^2 \cdot \|\cdot\|^2}{2}\Big),$$

and if in addition both $F_2 \circ \psi^*$ and $\breve{F}(\frac{\|\cdot\|^2}{2})$ are log-concave,

$$G^{orlicz}_{h,F_1,F_2}(\psi) \le G^{orlicz}_{h,\breve{F},\breve{F}}\Big(\frac{\bar{c}^2 \cdot \|\cdot\|^2}{2}\Big).$$

The above inequalities hold for $h \in \Psi$ with "$\le$" replaced by "$\ge$".

Moreover, if $h \in \Phi$ is strictly decreasing (or $h \in \Psi$ is strictly increasing) and $\breve{F}$ is strictly decreasing, equality holds if and only if $F_1, F_2, \breve{F}, \psi$ satisfy formula (11).

Proof. (i). First, $I(F_2 \circ \psi^*, \psi^*) = \hat{c}^{-n} I(\breve{F}, 1) = I(\breve{F}, \hat{c})$ by formula (8). Inequality (10) implies

$$I(F_1 \circ \psi, \psi) \le \hat{c}^n \cdot I(\breve{F}, 1) = I(\breve{F}, \hat{c}^{-1}).$$

Proposition 1 implies that for all $h \in \Phi$,

$$as^{orlicz}_{h,F_1,F_2}(\psi) \le I(F_1 \circ \psi, \psi) \cdot h\Big(\frac{(\sqrt{2\pi})^n}{I(F_2 \circ \psi^*, \psi^*)}\Big) \le I(\breve{F}, \hat{c}^{-1}) \cdot h\Big(\frac{(\sqrt{2\pi})^n}{\hat{c}^{-2n} \cdot I(\breve{F}, \hat{c}^{-1})}\Big), \tag{13}$$

and hence the desired result follows from Corollary 1.

Now let us characterize the condition for equality. First, assume that $F_1, F_2, \breve{F}, \psi$ satisfy formula (11). Letting $A = T^t T$ and $z = \sqrt{2} T y$, one has,

$$I\big(\breve{F}(\langle Ax, x\rangle)\big) = \int_{\mathbb{R}^n} \breve{F}\Big(\frac{\|\sqrt{2}Tx\|^2}{2}\Big)\, dx = \frac{1}{\sqrt{2^n \cdot \det A}} \int_{\mathbb{R}^n} \breve{F}\Big(\frac{\|z\|^2}{2}\Big)\, dz = \frac{I(\breve{F}, 1)}{\sqrt{2^n \cdot \det A}}.$$

Similar to the proof of Corollary 1, one can show that

$$\begin{aligned} as^{orlicz}_{h,F_1,F_2}(\psi) &= b \cdot I\big(\breve{F}(\langle Ax, x\rangle)\big) \cdot h\Bigg(\frac{b \cdot (\sqrt{2\pi})^n}{2^n \cdot \det A \cdot I\big(\breve{F}(\langle Ax, x\rangle)\big)}\Bigg) \\ &= \frac{b \cdot I(\breve{F}, 1)}{\sqrt{2^n \cdot \det A}} \cdot h\Bigg(\frac{b \cdot (\sqrt{2\pi})^n}{\sqrt{2^n \cdot \det A} \cdot I\big(\breve{F}, 1\big)}\Bigg). \end{aligned} \tag{14}$$

For $\psi(x) = \langle Ax, x\rangle + a$, one has $\psi^*(y) = \frac{1}{4}\langle A^{-1}y, y\rangle - a$ and hence,

$$\begin{aligned} I(F_2 \circ \psi^*, \psi^*) &= \frac{1}{b} \int_{\mathbb{R}^n} \breve{F}\Big(\frac{\langle A^{-1}y, y\rangle}{4}\Big)\, dy = \frac{\sqrt{2^n \cdot \det A}}{b}\ I(\breve{F}, 1), \\ \hat{c}^n &= \frac{I(\breve{F}, 1)}{I(F_2 \circ \psi^*, \psi^*)} = \frac{b}{\sqrt{2^n \cdot \det A}}. \end{aligned}$$

Corollary 1, formula (8) and formula (14) imply that if $F_1, F_2, \breve{F}, \psi$ satisfy formula (11),

$$\begin{aligned} as^{orlicz}_{h,\breve{F},\breve{F}}\Big(\frac{\|\cdot\|^2}{2\cdot\hat{c}^2}\Big) &= I(\breve{F},\hat{c}^{-1})\cdot h\Big(\frac{(\sqrt{2\pi})^n}{\hat{c}^{-2n}\cdot I(\breve{F},\hat{c}^{-1})}\Big) \\ &= \hat{c}^n I(\breve{F},1)\cdot h\Big(\frac{(\sqrt{2\pi})^n}{\hat{c}^{-n}\cdot I(\breve{F},1)}\Big) \\ &= \frac{b\cdot I(\breve{F},1)}{\sqrt{2^n\cdot \det A}}\cdot h\Big(\frac{b\cdot(\sqrt{2\pi})^n}{\sqrt{2^n\cdot\det A}\cdot I(\breve{F},1)}\Big) \\ &= as^{orlicz}_{h,F_1,F_2}(\psi). \end{aligned}$$

On the other hand, if $\breve{F}$ is strictly decreasing, then equality holds in (13) only if equality holds in inequality (10). That is, $F_1, F_2, \breve{F}, \psi$ satisfy formula (11). Hence, we have verified the desired characterization of equality in (i).

(ii). By inequality (10), one can check that $I(F_2\circ\psi^*,\psi^*)\leq \bar{c}^n I(\breve{F},1)$. Proposition 1 implies that for all decreasing $h\in\Phi$,

$$as^{orlicz}_{h,F_1,F_2}(\psi)\leq I(F_1\circ\psi,\psi)\cdot h\Big(\frac{(\sqrt{2\pi})^n}{I(F_2\circ\psi^*,\psi^*)}\Big)\leq I(\breve{F},\bar{c})\cdot h\Big(\frac{(\sqrt{2\pi})^n}{\bar{c}^{2n}\cdot I(\breve{F},\bar{c})}\Big),$$

and hence the desired result follows from Corollary 1. The characterization of equality follows along the same lines as in (i).

The desired results for $G^{orlicz}_{h,F_1,F_2}(\psi)$ follow along the same lines. The additional assumptions that both $F_2\circ\psi^*$ and $\breve{F}(\frac{\|\cdot\|^2}{2})$ are log-concave are needed in order to use Proposition 1 and Corollary 1. $\square$

The following result follows immediately from Theorem 3 by letting $F_1(t) = F_2(t) = e^{-t}$. These affine isoperimetric inequalities state that the maximum (minimum, respectively) of $as^{orlicz}_h(f)$ and $G^{orlicz}_h(f)$ for $h\in\Phi$ (for $h\in\Psi$, respectively) attain at (and only at) the Gaussian functions.

Corollary 2. *Let $\psi\in\mathcal{C}_0$ and $f=e^{-\psi}$.*

(i) For $h\in\Phi$, one has,

$$as^{orlicz}_h(f)\leq G^{orlicz}_h(f)\leq G^{orlicz}_h\Big(\exp\Big(-\frac{\|\cdot\|^2}{4\pi}\cdot\big[I\big(f^\circ\big)\big]^{\frac{2}{n}}\Big)\Big).$$

Equality holds if and only if ψ satisfies formula (12).

(ii) For decreasing $h\in\Phi$, one has,

$$as^{orlicz}_h(f)\leq G^{orlicz}_h(f)\leq G^{orlicz}_h\Big(\exp\Big(-\pi\|\cdot\|^2\cdot\big[I\big(f\big)\big]^{-\frac{2}{n}}\Big)\Big).$$

The above inequality holds for $h\in\Psi$ with "$\leq$" replaced by "$\geq$".

If $h\in\Phi$ is strictly decreasing (or $h\in\Psi$ is strictly increasing), equality holds if and only if ψ satisfies formula (12).

We now establish cyclic inequalities for the general Orlicz affine and geominimal surface areas of convex functions. Assume that the function h_1 always has inverse h_1^{-1} and let $H = h \circ h_1^{-1}$. Moreover, $H(0)$ and $H(\infty)$ are defined by the limit of $H(t)$ as $t \to 0$ and $t \to \infty$ respectively (could be a finite number or ∞, if exist). Note that if h^{-1} and h_1^{-1} both exist, then condition (a) is equivalent to condition (d); and condition (c) is equivalent to condition (f) if in addition H is increasing.

Theorem 4. *Let $F_1, F_2 \colon \mathbb{R} \to (0, \infty)$ be measurable functions and $\psi \in \mathcal{C}$.*

(i) Assume one of the following conditions: (a) $h \in \Phi$ and $h_1 \in \Psi$ with H increasing; (b) $h, h_1 \in \Phi$ with H decreasing; (c) H concave increasing with either $h, h_1 \in \Phi$ or $h, h_1 \in \Psi$. Then

$$\frac{as^{orlicz}_{h,F_1,F_2}(\psi)}{I(F_1 \circ \psi, \psi)} \leq H\bigg(\frac{as^{orlicz}_{h_1,F_1,F_2}(\psi)}{I(F_1 \circ \psi, \psi)}\bigg).$$

(ii) Assume one of the following conditions: (d) $h \in \Psi$ and $h_1 \in \Phi$ with H increasing; (e) H convex decreasing with one in Φ and the other one in Ψ; (f) H convex increasing with either $h, h_1 \in \Phi$ or $h, h_2 \in \Psi$. Then

$$\frac{as^{orlicz}_{h,F_1,F_2}(\psi)}{I(F_1 \circ \psi, \psi)} \geq H\bigg(\frac{as^{orlicz}_{h_1,F_1,F_2}(\psi)}{I(F_1 \circ \psi, \psi)}\bigg).$$

The same inequalities also hold for the general Orlicz geominimal surface area of convex functions, if in addition $F_2 \circ \psi^* \in \mathcal{L}_{\psi^*}$ in conditions (a), (b) and (d).

Remark. In particular, the above inequalities hold for $as^{orlicz}_h(f)$ and $G^{orlicz}_h(f)$, as long as corresponding conditions are verified.

Proof. For completeness, we include a brief proof, which is similar to that of Theorem 3.1 in [36]. Results for conditions (a), (b) and (d) follow immediately from Proposition 1 and the monotonicity of H. Results for conditions (c), (e) and (f) hold by the combination of Jensen's inequality, the monotonicity of H, and Definition 3. Here, as an example, we show the case for condition (c) and omit the proofs for other cases. Jensen's inequality to the concave function H implies

$$\frac{V_{h,F_1,F_2}(\psi, g)}{I(F_1 \circ \psi, \psi)} \leq H\bigg(\frac{V_{h_1,F_1,F_2}(\psi, g)}{I(F_1 \circ \psi, \psi)}\bigg).$$

As H is increasing and $h, h_1 \in \Phi$, then

$$\begin{aligned}
\frac{as^{orlicz}_{h,F_1,F_2}(\psi)}{I(F_1 \circ \psi, \psi)} &= \inf\bigg\{\frac{V_{h,F_1,F_2}(\psi, g)}{I(F_1 \circ \psi, \psi)} : g \in \mathcal{F}^+_{\psi^*} \text{ with } I(g, \psi^*) = (\sqrt{2\pi})^n\bigg\} \\
&\leq H\bigg(\inf\bigg\{\frac{V_{h_1,F_1,F_2}(\psi, g)}{I(F_1 \circ \psi, \psi)} : g \in \mathcal{F}^+_{\psi^*} \text{ with } I(g, \psi^*) = (\sqrt{2\pi})^n\bigg\}\bigg) \\
&= H\bigg(\frac{as^{orlicz}_{h_1,F_1,F_2}(\psi)}{I(F_1 \circ \psi, \psi)}\bigg).
\end{aligned}$$

The case $h, h_1 \in \Psi$ follows similarly with " inf " replaced by " sup". $\square$

2.4 The general L_p geominimal surface area for convex functions and a Santaló type inequality

Theorem 1 and Definition 3 yield that

$$as_{p,F_1,F_2}(\psi) = (\sqrt{2\pi})^{\frac{np}{n+p}} \left[as^{orlicz}_{h,F_1,F_2}(\psi)\right]^{\frac{n}{n+p}}$$

with $h(t) = t^{-p/n}$ for $-n \neq p \in \mathbb{R}$. Its properties have been discussed in e.g. [10, 11].

In this subsection, we briefly discuss properties for the general L_p geominimal surface areas of convex functions for $-n \neq p \in \mathbb{R}$. Taking Theorem 1 into account, it is more natural to define the general L_p geominimal surface areas of convex functions as

$$G_{p,F_1,F_2}(\psi) = (\sqrt{2\pi})^{\frac{np}{n+p}} \left[G^{orlicz}_{h,F_1,F_2}(\psi)\right]^{\frac{n}{n+p}}$$

with $h(t) = t^{-p/n}$ for $-n \neq p \in \mathbb{R}$.

Definition 4. *For $p \geq 0$, define the general L_p geominimal surface area of $\psi \in \mathcal{C}$ by*

$$G_{p,F_1,F_2}(\psi) = \inf_{g \in \mathcal{L}_{\psi^*}} \left\{ V_{p,F_1,F_2}(\psi, g)^{\frac{n}{n+p}} \ I(g, \psi^*)^{\frac{p}{n+p}} \right\};$$

while for $-n \neq p < 0$, $G_{p,F_1,F_2}(\psi)$ is defined similarly but with "$\inf$" replaced by "$\sup$".

In particular, the L_p geominimal surface area of $f = e^{-\psi}$ can be defined as

$$G_p(f) = G_{p,e^{-t},e^{-t}}(\psi).$$

Results in subsections 2.2 and 2.3 can be modified accordingly to $G_{p,F_1,F_2}(\psi)$ and $G_p(f)$. For instance, $G_{p,F_1,F_2}(\psi)$ is $SL_{\pm}(n)$-invariant. Moreover, for $T \in GL(n)$,

$$G_{p,F_1,F_2}(\psi \circ T) = \left|\det(T)\right|^{\frac{p-n}{p+n}} \cdot G_{p,F_1,F_2}(\psi).$$

It also has the homogeneous degree $\frac{n(p-n)}{p+n}$, i.e.,

$$G_{p,F_1,F_2}(\psi \circ \lambda) = |\lambda|^{\frac{n(p-n)}{p+n}} \cdot G_{p,F_1,F_2}(\psi),$$

where $(\psi \circ \lambda)(x) = \psi(\lambda x)$ for $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$.

Let $F \colon \mathbb{R} \to (0, \infty)$ satisfy that $F\big(\frac{\|\cdot\|^2}{2}\big)$ is log-concave and $0 < I(F, 1) < \infty$. Corollary 1 implies that for $-n \neq p \in \mathbb{R}$ and $c > 0$ a constant,

$$G_{p,F,F}\Big(\frac{c^2\|\cdot\|^2}{2}\Big) = c^{\frac{n(p-n)}{n+p}} \cdot I\big(F, 1\big), \tag{15}$$

and in particular $G_p\big(\gamma_n \circ c\big) = c^{\frac{n(p-n)}{p+n}} (\sqrt{2\pi})^n$.

A direct consequence of Proposition 1 is the following result. Similar inequalities were obtained in [10, 11].

Proposition 2. *Let $\psi \in \mathcal{C}$. If $F_2 \circ \psi^* \in \mathcal{L}_{\psi^*}$, then for $p \in (0,\infty)$ and $f = e^{-\psi}$,*

$$\begin{aligned} G_{p,F_1,F_2}(\psi) &\le \big[I(F_1 \circ \psi, \psi)\big]^{\frac{n}{n+p}} \cdot \big[I(F_2 \circ \psi^*, \psi^*)\big]^{\frac{p}{n+p}}, \\ G_p(f) &\le \big[I(f)\big]^{\frac{n}{n+p}} \cdot \big[I(f^\circ)\big]^{\frac{p}{n+p}}. \end{aligned}$$

Similar inequalities hold for $p \in (-\infty,-n) \cup (-n,0)$ with "$\le$" replaced by "$\ge$".

Immediately from Theorem 3, one has the following functional L_p affine isoperimetric inequalities.

Theorem 5. *Let $F_1, F_2 \colon \mathbb{R} \to (0,\infty)$ be measurable functions and $\psi \in \mathcal{C}_0$ such that $F_2 \circ \psi^*$ and $\breve{F}\big(\frac{\|\cdot\|^2}{2}\big)$ are log-concave. Assume that $0 < I(\breve{F},1) < \infty$.*

(i) Let $p > 0$. Then,

$$\frac{G_{p,F_1,F_2}(\psi)}{G_{p,\breve{F},\breve{F}}\big(\frac{\|\cdot\|^2}{2}\big)} \le \min\left\{ \left(\frac{I(F_2 \circ \psi^*, \psi^*)}{I(\breve{F},1)}\right)^{\frac{p-n}{p+n}}, \ \left(\frac{I(F_1 \circ \psi, \psi)}{I(\breve{F},1)}\right)^{\frac{n-p}{n+p}} \right\}.$$

(ii) Let $p \in (-n,0)$. Then,

$$\frac{G_{p,F_1,F_2}(\psi)}{G_{p,\breve{F},\breve{F}}\big(\frac{\|\cdot\|^2}{2}\big)} \ge \left(\frac{I(F_1 \circ \psi, \psi)}{I(\breve{F},1)}\right)^{\frac{n-p}{n+p}}.$$

(iii) Let $p < -n$. Then,

$$\frac{G_{p,F_1,F_2}(\psi)}{G_{p,\breve{F},\breve{F}}\big(\frac{\|\cdot\|^2}{2}\big)} \ge \left(\frac{I(F_2 \circ \psi^*, \psi^*)}{I(\breve{F},1)}\right)^{\frac{p-n}{p+n}}.$$

If $\breve{F}$ is strictly decreasing, equality holds in each case if and only if $\psi, \breve{F}, F_1$ and F_2 satisfy formula (11).

In particular, for the L_p geominimal surface area of log-concave functions, one has the following functional L_p affine isoperimetric inequality. Similar inequalities were obtained in [10, 11].

Corollary 3. *Let $\psi \in \mathcal{C}_0$ and $f = e^{-\psi}$.*

(i) Let $p > 0$. Then,

$$\frac{G_p(f)}{G_p(\gamma_n)} \le \min\left\{ \left(\frac{I(f^\circ)}{I(\gamma_n)}\right)^{\frac{p-n}{p+n}}, \ \left(\frac{I(f)}{I(\gamma_n)}\right)^{\frac{n-p}{n+p}} \right\}.$$

(ii) Let $p \in (-n,0)$. Then,

$$\frac{G_p(f)}{G_p(\gamma_n)} \ge \left(\frac{I(f)}{I(\gamma_n)}\right)^{\frac{n-p}{n+p}}.$$

(iii) Let $p < -n$. Then,

$$\frac{G_p(f)}{G_p(\gamma_n)} \ge \left(\frac{I(f^\circ)}{I(\gamma_n)}\right)^{\frac{p-n}{p+n}}.$$

Equality holds in each case if and only if ψ satisfies formula (12).

In the following theorem, we provide a Santaló type inequality for the general L_p geominimal surface area of convex functions. It is a generalization of inequality (10).

Theorem 6. *Let $F_1, F_2 \colon \mathbb{R} \to (0,\infty)$ be measurable functions and $\psi \in \mathcal{C}_0$ such that $F_1 \circ \psi$, $F_2 \circ \psi^*$ and $\breve{F}\big(\frac{\|\cdot\|^2}{2}\big)$ are log-concave. Assume that $0 < I(\breve{F}, 1) < \infty$. Then, for $p > 0$,*

$$G_{p,F_1,F_2}(\psi) \cdot G_{p,F_2,F_1}(\psi^*) \leq \Big[G_{p,\breve{F},\breve{F}}\Big(\frac{\|\cdot\|^2}{2}\Big)\Big]^2.$$

If $\breve{F}$ is strictly decreasing, equality holds if and only if $\psi, \breve{F}, F_1$ and F_2 satisfy formula (11).

Proof. For $p > 0$, by Proposition 2 and inequality (10), one has,

$$\begin{aligned} G_{p,F_1,F_2}(\psi) \cdot G_{p,F_2,F_1}(\psi^*) &\leq I(F_1 \circ \psi, \psi) \cdot I(F_2 \circ \psi^*, \psi^*) \\ &\leq \big[I(\breve{F}, 1)\big]^2 = \Big[G_{p,\breve{F},\breve{F}}\Big(\frac{\|\cdot\|^2}{2}\Big)\Big]^2, \end{aligned}$$

where the last equality follows from formula (15). The characterization of equality follows along the same lines as in Theorem 3. □

More generally, if $h \in \Phi$ such that $h(t)h(s) \leq [h(r)]^2$ for all $r, s, t > 0$ satisfying $st \geq r^2$, then

$$as^{orlicz}_{h,F_1,F_2}(\psi) \cdot as^{orlicz}_{h,F_2,F_1}(\psi^*) \leq \Big[as^{orlicz}_{h,\breve{F},\breve{F}}\Big(\frac{\|\cdot\|^2}{2}\Big)\Big]^2,$$

and if in addition $F_1 \circ \psi$, $F_2 \circ \psi^*$ and $\breve{F}\big(\frac{\|\cdot\|^2}{2}\big)$ are log-concave,

$$G^{orlicz}_{h,F_1,F_2}(\psi) \cdot G^{orlicz}_{h,F_2,F_1}(\psi^*) \leq \Big[G^{orlicz}_{h,\breve{F},\breve{F}}\Big(\frac{\|\cdot\|^2}{2}\Big)\Big]^2.$$

Moreover, the following Santaló type inequality for log-concave functions holds. These results extend the functional Blaschke-Santaló and inverse Santaló inequality [2, 5, 13, 14, 17, 18]. Similar inequalities were obtained in [10, 11].

Corollary 4. *Let $\psi \in \mathcal{C}_0$ and $f = e^{-\psi}$.*

(i) For $p \in (0, \infty)$, the following inequality holds, with equality if and only if ψ satisfies formula (12),

$$G_p(f) \cdot G_p(f^\circ) \leq \big[G_p(\gamma_n)\big]^2 = (2\pi)^n.$$

(ii) For $p \in (-\infty, -n) \cup (-n, 0)$, there is a universal constant $C > 0$, such that,

$$G_p(f) \cdot G_p(f^\circ) \geq C^n \cdot \big[G_p(\gamma_n)\big]^2.$$

Proof. The part (i) follows immediately from Theorem 6. Now let $p \in (-\infty, -n) \cup (-n, 0)$. By Proposition 2, one has,

$$G_p(f) \cdot G_p(f^\circ) \geq I(f) \cdot I(f^\circ) \geq C^n \cdot (2\pi)^n = C^n \cdot \big[G_p(\gamma_n)\big]^2,$$

where the second inequality follows from the functional inverse Santaló inequality [14, 17]. □

Remark. Let $h \in \Phi$ be such that $h(t)h(s) \le [h(r)]^2$ for all $r, s, t > 0$ satisfying $st \ge r^2$, then

$$G_h^{orlicz}(f) \cdot G_h^{orlicz}(f^\circ) \le \big[G_h^{orlicz}(\gamma_n)\big]^2;$$

while, if $h \in \Psi$ satisfying $h(t)h(s) \ge A \cdot [h(r)]^2$ for some constant $A > 0$ and for all $r, s, t > 0$ satisfying $st \ge r^2$, then, there is a universal constant $C > 0$, such that

$$G_h^{orlicz}(f) \cdot G_h^{orlicz}(f^\circ) \ge AC^n \cdot \big[G_h^{orlicz}(\gamma_n)\big]^2.$$

3 Orlicz affine and geominimal surface areas for s-concave functions

Let $s \in (0, \infty)$. A nonnegative function f is s-concave if f^s is concave on its support [7], that is, for all $\lambda \in [0, 1]$ and for all $x, y \in \mathbb{R}^n$ such that $f(x) > 0$ and $f(y) > 0$, one has,

$$f((1-\lambda)x + \lambda y) \ge \big((1-\lambda)f(x)^s + \lambda f(y)^s\big)^{1/s}.$$

The support set of f is $S_f := \{x : f(x) > 0\}$. Note that S_f is a convex set in $\mathbb{R}^n$. Throughout this section, assume that S_f is open and bounded with $0 \in \mathbb{R}^n$ in the interior of S_f, and $\lim_{x \to \partial S_f} f(x) = 0$ where ∂S_f is the boundary of S_f. Let $\mathcal{C}_s$ be the collection of all upper semi-continuous s-concave functions whose supports satisfy above assumptions. Define the function ψ on S_f by

$$\psi(x) = \frac{1 - f^s(x)}{s} \quad \Leftrightarrow \quad f(x) = \big(1 - s\psi(x)\big)^{\frac{1}{s}}, \quad \forall x \in S_f. \tag{16}$$

Note that ψ is well defined and is convex on S_f. Moreover, $1 - s\psi = f^s > 0$ and hence $\psi(x) < \frac{1}{s}$ for all $x \in S_f$. In the later context, the pair of functions (f, ψ) refers to $f \in \mathcal{C}_s$ and its associated convex function ψ by formula (16). The following dual function $\psi^\star_{(s)}$ for convex function ψ is crucial in this section:

$$\psi^\star_{(s)}(y) = \sup_{x \in S_f} \frac{\langle x, y\rangle - \psi(x)}{1 - s\psi(x)}. \tag{17}$$

It is easily checked that $\psi^\star_{(s)}$ is convex and $(\psi^\star_{(s)})^\star_{(s)} = \psi$ for all $f \in \mathcal{C}_s$. With the help of $\psi^\star_{(s)}$, one can define the (s)-Legendre dual of $f \in \mathcal{C}_s$ by

$$f^\circ_{(s)}(y) = \big(1 - s\psi^\star_{(s)}(y)\big)^{1/s}, \qquad \forall y \in S_{f^\circ_{(s)}} = \big\{y : 1 - s\psi^\star_{(s)}(y) > 0\big\}.$$

Equivalently (which coincides with the definition introduced in [2, 4]), by letting $a_+ = \max\{a, 0\}$,

$$f^\circ_{(s)}(y) = \inf_{x \in S_f} \frac{\big[(1 - s\langle x, y\rangle)_+\big]^{1/s}}{f(x)}.$$

Note that $f^\circ_{(s)}$ is s-concave and upper semi-continuous. Moreover, $(f^\circ_{(s)})^\circ_{(s)} = f$ and $S_{f^\circ_{(s)}} = \frac{1}{s} S_f^\circ$ where

$$S_f^\circ = \{z : \langle x, z\rangle < 1 \quad \text{for all } x \in S_f\}.$$

Throughout this section, let $X_\psi \subset S_f$ be such that

$$X_\psi = \Big\{x \in S_f : \nabla^2\psi, \text{ the Hessian matrix of } \psi \text{ in the sense of Alexandrov, exists and is invertible}\Big\}.$$

For simplicity, let $\widetilde{\psi}(x) = 1 + s\langle x, \nabla\psi(x)\rangle - s\psi(x)$. The supremum in (17) is attained if $x \in S_f$ and

$$y = \frac{1 - s\langle x, y\rangle}{1 - s\psi(x)} \nabla\psi(x) \quad \text{or} \quad y = (1 - s\psi^\star_{(s)}(y))\nabla\psi(x).$$

This leads to $\langle x, y\rangle = \frac{1-s\langle x,y\rangle}{1-s\psi(x)} \langle x, \nabla\psi(x)\rangle$ and

$$\frac{1}{1 - s\psi^\star_{(s)}(y)} = \frac{1 - s\psi(x)}{1 - s\langle x, y\rangle} = 1 + s\langle x, \nabla\psi(x)\rangle - s\psi(x) = \widetilde{\psi}(x). \tag{18}$$

That is, the supremum in (17) is attained if $x \in S_f$ and $y = \frac{\nabla\psi(x)}{\widetilde{\psi}(x)} = T_\psi(x)$. Moreover,

$$\psi^\star_{(s)}(y) = \frac{\langle x, y\rangle - \psi(x)}{1 - s\psi(x)}, \qquad dy = \frac{1 - s\psi(x)}{\big(\widetilde{\psi}(x)\big)^{n+1}} \det\nabla^2\psi(x)\, dx. \tag{19}$$

See [10] for details. Similar to the proof of Theorem 4 in [10], for an integrable function g defined on $X_{\psi^\star_{(s)}}$, one has,

$$I_s(g, \psi^\star_{(s)}) = \int_{X_{\psi^\star_{(s)}}} g(y)\, dy = \int_{X_\psi} g\big(T_\psi(x)\big) \cdot \frac{(1 - s\psi(x)) \cdot \det\nabla^2\psi(x)}{(\widetilde{\psi}(x))^{n+1}}\, dx. \tag{20}$$

3.1 Definition and Properties

Let $h : (0, \infty) \to (0, \infty)$ be a continuous function. For simplicity, let $\mathcal{F}^+_{s,\star}$ be the set of all positive integrable functions defined on $X_{\psi^\star_{(s)}}$, i.e., for all $g \in \mathcal{F}^+_{s,\star}$, one has $g(y) > 0$ for all $y \in X_{\psi^\star_{(s)}}$ and $0 < I_s(g, \psi^\star_{(s)}) < \infty$. Let (f, ψ) be the pair given by formula (16) and $f \in \mathcal{C}_s$.

Definition 5. *The Orlicz $L^{(s)}_h$-mixed integral of ψ and $g \in \mathcal{F}^+_{s,\star}$ is defined by*

$$V^{(s)}_h(\psi, g) = \int_{X_\psi} h\Big(g(T_\psi(x))(\widetilde{\psi}(x))^{(\frac{1}{s}-1)}(1 - s\psi(x))\Big) \cdot \widetilde{\psi}(x) \cdot (1 - s\psi(x))^{(\frac{1}{s}-1)}\, dx.$$

It can be proved, similar to the proof of Lemma 1, that for all $T \in SL_\pm(n)$, one has,

$$V^{(s)}_h(\psi \circ T, g \circ T^{-t}) = V^{(s)}_h(\psi, g).$$

We write $V^{(s)}_p(\psi, g)$ for the case $h(t) = t^{-p/n}$ with $-n \neq p \in \mathbb{R}$.

The following definition for the L_p affine surface area of s-concave functions was given in [10].

Definition 6. *Let $s > 0$ and the pair (f, ψ) be given by formula (16). For any $-n \neq p \in \mathbb{R}$, the L_p affine surface area of the s-concave function f is defined by*

$$as^{(s)}_p(\psi) = \frac{1}{1 + ns} \int_{X_\psi} \frac{(1 - s\psi(x))^{\left(\frac{1}{s}-1\right)\cdot\frac{n}{n+p}} \left(\det\nabla^2\psi(x)\right)^{\frac{p}{n+p}}}{\left(\widetilde{\psi}(x)\right)^{\frac{p}{n+p}\left(n+\frac{1}{s}+1\right)-1}}\, dx.$$

The following theorem provides a new formula for $as_p^{(s)}(\psi)$.

Theorem 7. *Let $s>0$ and the pair (f,ψ) be given by formula (16) with $f\in\mathcal{C}_s$. Assume that ψ is a C^2 strictly convex function. For $p\geq 0$, one has*

$$as_p^{(s)}(\psi)=\frac{1}{1+ns}\inf_{g\in\mathcal{F}^+_{s,\star}}\left\{V_p^{(s)}(\psi,g)^{\frac{n}{n+p}}\ I_s(g,\psi^\star_{(s)})^{\frac{p}{n+p}}\right\},$$

while for $-n\neq p<0$, the above formula holds with "$\inf$" replaced by "$\sup$".

Proof. We only prove the case for $p\geq 0$ and the case for $-n\neq p<0$ follows similarly by the (reverse) Hölder's inequality. It is clear that for $p=0$ and for all $g\in\mathcal{F}^+_{s,\star}$ (see [10] for details),

$$V_0^{(s)}(\psi,g)=\int_{X_\psi}(1-s\psi(x))^{(\frac{1}{s}-1)}\,\widetilde{\psi}(x)\,dx=(1+ns)\cdot as_0^{(s)}(\psi).$$

Let $p\in(0,\infty)$ and thus $\frac{p}{n+p}\in(0,1)$. Hölder's inequality implies that for all function $g\in\mathcal{F}^+_{s,\star}$,

$$\begin{aligned} as_p^{(s)}(\psi) &= \frac{1}{1+ns}\int_{X_\psi}\left[\left(\frac{g^{-1}(T_\psi(x))}{(\widetilde{\psi}(x))^{(\frac{1}{s}-1)}(1-s\psi(x))}\right)^{\frac{p}{n}}\cdot\widetilde{\psi}(x)\cdot(1-s\psi(x))^{(\frac{1}{s}-1)}\right]^{\frac{n}{n+p}} \\ &\quad\times\left[g(T_\psi(x))\cdot\frac{(1-s\psi(x))\cdot\det\nabla^2\psi(x)}{(\widetilde{\psi}(x))^{n+1}}\right]^{\frac{p}{n+p}}\ dx \\ &\leq \frac{1}{1+ns}V_p^{(s)}(\psi,g)^{\frac{n}{n+p}}\ I_s(g,\psi^\star_{(s)})^{\frac{p}{n+p}}. \end{aligned}$$

Taking the infimum over all $g\in\mathcal{F}^+_{s,\star}$, one gets, for all $p\in(0,\infty)$,

$$as_p^{(s)}(\psi)\leq\frac{1}{1+ns}\inf_{g\in\mathcal{F}^+_{s,\star}}\left\{V_p^{(s)}(\psi,g)^{\frac{n}{n+p}}\ I_s(g,\psi^\star_{(s)})^{\frac{p}{n+p}}\right\}.$$

Let g_0 be the function given by

$$g_0(T_\psi(x))=\left(\frac{(1-s\psi(x))^{(\frac{1}{s}-2-\frac{p}{n})}\cdot(\widetilde{\psi}(x))^{(n+2-\frac{p}{ns}+\frac{p}{n})}}{\det\nabla^2\psi(x)}\right)^{\frac{n}{n+p}}.$$

Then $I_s(g_0,\psi^\star_{(s)})=(1+ns)as_p^{(s)}(\psi)=V_p^{(s)}(\psi,g_0)$ (see also Theorem 4 in [10]) and

$$\begin{aligned} as_p^{(s)}(\psi) &= \frac{1}{1+ns}\left\{V_p^{(s)}(\psi,g_0)^{\frac{n}{n+p}}\ I_s(g_0,\psi^\star_{(s)})^{\frac{p}{n+p}}\right\} \\ &\geq \frac{1}{1+ns}\inf_{g\in\mathcal{F}^+_{s,\star}}\left\{V_p^{(s)}(\psi,g)^{\frac{n}{n+p}}\ I_s(g,\psi^\star_{(s)})^{\frac{p}{n+p}}\right\}, \end{aligned}$$

Thus the desired result follows.

Now let us find an explicit expression for g_0. Recall that $(\psi^\star_{(s)})^\star_{(s)}=\psi$, which implies $T_\psi\circ T_{\psi^\star_{(s)}}=\mathrm{Id}$ and $T_{\psi^\star_{(s)}}\circ T_\psi=\mathrm{Id}$. Hence, for $x\in X_\psi$ and $y=T_\psi(x)$, one has

$$\det\,(d_xT_\psi)\cdot\det\left(d_yT_{\psi^\star_{(s)}}\right)=1. \tag{21}$$

Moreover, for $x \in X_\psi$ and $y = T_\psi(x)$, equation (18) implies

$$\frac{1}{1 - s\psi^\star_{(s)}(y)} = \widetilde{\psi}(x) \quad \text{and} \quad \frac{1}{1 - s\psi(x)} = 1 + s(\langle \nabla \psi^\star_{(s)}(y), y\rangle - \psi^\star_{(s)}(y)) = \widetilde{\psi^\star_{(s)}}(y). \tag{22}$$

Combining (19) with (21), one gets

$$\det \nabla^2 \psi(x) \bigg(\frac{1 - s\psi^\star_{(s)}(y)}{1 + s\langle \nabla \psi^\star_{(s)}(y), y\rangle - s\psi^\star_{(s)}(y)} \bigg)^{n+2} \det \nabla^2 \psi^\star_{(s)}(y) = 1.$$

Thus, if $y = T_\psi(x)$, then

$$g_0(y) \quad = \quad \bigg(\frac{(1 + s\langle \nabla \psi^\star_{(s)}(y), y\rangle - s\psi^\star_{(s)}(y))^{(-\frac{1}{s} + \frac{p}{n} - n)}}{(1 - s\psi^\star_{(s)}(y))^{(\frac{p}{n} - \frac{p}{ns})}} \bigg)^{\frac{n}{n+p}} \ (\det \nabla^2 \psi^\star_{(s)}(y))^{\frac{n}{n+p}}.$$

□

For $s > 0$, let $k_s(x) = \big[\big(1 - s\|x\|^2\big)_+\big]^{\frac{1}{2s}}$. Note that $k_s(\cdot)$ is the special function which plays the role of the unit "Euclidean ball" in s-concave functions, that is, $(k_s)^\circ_{(s)} = k_s$ (see e.g., [3]). We also let

$$\int_{\{x \in \mathbb{R}^n : \|x\| < s^{-1/2}\}} k_s(x)\, dx = \Big(\frac{\pi}{s}\Big)^{\frac{n}{2}} \frac{\Gamma(1 + \frac{1}{2s})}{\Gamma(1 + \frac{n}{2} + \frac{1}{2s})} =: \omega_{n,s}.$$

Motivated by Theorem 7, we now propose the following definition for the Orlicz affine and geominimal surface areas for s-concave functions. Let $\mathcal{L}_{s,\star} \subset \mathcal{F}^+_{s,\star}$ be the subset containing all log-concave functions.

Definition 7. *Let (f, ψ) be the pair given by formula (16) with $f \in \mathcal{C}_s$. For $h \in \Phi$, the Orlicz $L_h^{(s)}$ affine surface area of ψ is defined by*

$$as^{orlicz}_{h,s}(\psi) = \inf \Big\{ V_h^{(s)}(\psi, g) : g \in \mathcal{F}^+_{s,\star} \text{ with } I_s(g, \psi^\star_{(s)}) = (1 + ns) \cdot \omega_{n,s} \Big\},$$

and the Orlicz $L_h^{(s)}$ geominimal surface area of ψ is defined by

$$G^{orlicz}_{h,s}(\psi) = \inf \Big\{ V_h^{(s)}(\psi, g) : g \in \mathcal{L}_{s,\star} \text{ with } I_s(g, \psi^\star_{(s)}) = (1 + ns) \cdot \omega_{n,s} \Big\}.$$

When $h \in \Psi$, the Orlicz $L_h^{(s)}$ affine and geominimal surface areas of ψ are defined as above with " $\inf$" replaced by " $\sup$".

One can easily see that both $as^{orlicz}_{h,s}(\psi)$ and $G^{orlicz}_{h,s}(\psi)$ are $SL_\pm(n)$-invariant in the same fashion of Theorem 2. It is clear that $as^{orlicz}_{h,s}(\psi) \le G^{orlicz}_{h,s}(\psi)$ for $h \in \Phi$ and $as^{orlicz}_{h,s}(\psi) \ge G^{orlicz}_{h,s}(\psi)$ for $h \in \Psi$.

Hereafter, for a constant $c > 0$, let $\mathcal{E}^s_c(x) = \frac{1 - [(1 - sc^2\|x\|^2)_+]^{1/2}}{s}$. It can be checked that $(\mathcal{E}^s_c)^\star = \mathcal{E}^s_{1/c}$. The function $\mathcal{E}^s_c$ is associated to the s-concave function $k^c_s(x) = \big[(1 - sc^2\|x\|^2)_+\big]^{\frac{1}{2s}}$ by identity (16). Note that $X_{\mathcal{E}^s_c} = \{x : \|x\| < c^{-1}s^{-1/2}\}$ and $X_{(\mathcal{E}^s_c)^\star} = \{y : \|y\| < cs^{-1/2}\}$. Moreover,

$$I_s(k^c_s) = \int_{X_{\mathcal{E}^s_c}} k^c_s(x)\, dx = c^{-n} \cdot \omega_{n,s}.$$

Applying identity (24) (which will be stated in the next subsection and was proved in [10]) to function $\mathcal{E}_c^s$, one has,

$$(1+ns)\cdot I_s(k_s^c) = \int_{X_{\mathcal{E}_c^s}} \widetilde{\mathcal{E}_c^s}(x)\cdot\big(1-s\mathcal{E}_c^s(x)\big)^{(\frac{1}{s}-1)}\,dx = \int_{X_{\mathcal{E}_c^s}} \big(1-sc^2\|x\|^2\big)_+^{(\frac{1}{2s}-1)}\,dx. \tag{23}$$

The following corollary provides a precise value for the Orlicz affine and geominimal surface areas of k_s^c.

Corollary 5. *Let $c>0$ be a constant. For all $h\in\Phi\cup\Psi$,*

$$as_{h,s}^{orlicz}\big(\mathcal{E}_c^s\big) = (1+ns)\cdot I_s\big(k_s^c\big)\cdot h(c^{-n}) = (1+ns)\cdot c^{-n}\cdot\omega_{n,s}\cdot h(c^{-n}),$$

and if in addition $\Big[\big(1-\frac{s\|\cdot\|^2}{c^2}\big)_+\Big]^{(\frac{1}{2s}-1)}$ is a log-concave function (which holds if $s\le 1/2$),

$$G_{h,s}^{orlicz}\big(\mathcal{E}_c^s\big) = (1+ns)\cdot I_s\big(k_s^c\big)\cdot h(c^{-n}) = (1+ns)\cdot c^{-n}\cdot\omega_{n,s}\cdot h(c^{-n}).$$

Proof. Note that $\nabla\mathcal{E}_c^s(x) = \frac{c^2x}{\big[(1-sc^2\|x\|^2)_+\big]^{1/2}}$ and $\big(1-s\mathcal{E}_c^s(x)\big) = \big[(1-sc^2\|x\|^2)_+\big]^{1/2}$. Moreover,

$$\widetilde{\mathcal{E}_c^s}(x) = 1+s\langle x,\nabla\mathcal{E}_c^s(x)\rangle - s\mathcal{E}_c^s(x) = \big[(1-sc^2\|x\|^2)_+\big]^{-1/2} \qquad \text{for } x\in X_{\mathcal{E}_c^s},$$

which leads to $T_{\mathcal{E}_c^s}(x) = \frac{\nabla\mathcal{E}_c^s(x)}{\widetilde{\mathcal{E}_c^s}(x)} = c^2x$.

Applying Jensen's inequality to the convex function h (as $h\in\Phi$) and by formula (23), one has,

$$\begin{aligned}
V_h^{(s)}(\mathcal{E}_c^s,g) &= \int_{X_{\mathcal{E}_c^s}} h\Big(g(T_{\mathcal{E}_c^s}(x))(\widetilde{\mathcal{E}_c^s}(x))^{(\frac{1}{s}-1)}\big(1-s\mathcal{E}_c^s(x)\big)\Big)\cdot\widetilde{\mathcal{E}_c^s}(x)\cdot(1-s\mathcal{E}_c^s(x))^{(\frac{1}{s}-1)}\,dx\\
&\ge (1+ns)\cdot I_s(k_s^c)\cdot h\bigg(\int_{X_{\mathcal{E}_c^s}} \frac{g(c^2x)\cdot(\widetilde{\mathcal{E}_c^s}(x))^{\frac{1}{s}}\cdot(1-s\mathcal{E}_c^s(x))^{\frac{1}{s}}}{(1+ns)\cdot I_s(k_s^c)}\,dx\bigg)\\
&= (1+ns)\cdot I_s(k_s^c)\cdot h\bigg(\int_{X_{(\mathcal{E}_c^s)^\star}} \frac{g(y)}{c^{2n}\cdot(1+ns)\cdot I_s(k_s^c)}\,dy\bigg)\\
&= (1+ns)\cdot I_s(k_s^c)\cdot h\bigg(\frac{I_s(g,\psi_{(s)}^\star)}{c^{2n}(1+ns)\cdot I_s(k_s^c)}\bigg).
\end{aligned}$$

This leads to, for all $h\in\Phi$,

$$\begin{aligned}
as_{h,s}^{orlicz}\big(\mathcal{E}_c^s\big) &= \inf\Big\{V_h^{(s)}(\mathcal{E}_c^s,g) : g\in\mathcal{F}_{s,\star}^+ \text{ with } I_s(g,\psi_{(s)}^\star) = (1+ns)\cdot\omega_{n,s}\Big\}\\
&\ge (1+ns)\cdot I_s\big(k_s^c\big)\cdot h(c^{-n}).
\end{aligned}$$

On the other hand,

$$as_{h,s}^{orlicz}\big(\mathcal{E}_c^s\big) \le V_h^{(s)}\bigg(\mathcal{E}_c^s, c^{-n}\cdot\Big[\Big(1-\frac{s\|\cdot\|^2}{c^2}\Big)_+\Big]^{(\frac{1}{2s}-1)}\bigg) = (1+ns)\cdot I_s\big(k_s^c\big)\cdot h(c^{-n}),$$

where we have used identity (23)

$$\int_{\{x\in\mathbb{R}^n:\ \|x\|\le cs^{-1/2}\}} \Big[\Big(1-\frac{s\|x\|^2}{c^2}\Big)_+\Big]^{(\frac{1}{2s}-1)}\,dx = c^n\cdot\omega_{n,s}\cdot(1+ns).$$

Hence, the desired formula follows. Along the same lines, one gets the desired formula for $h \in \Psi$.

The proof of the geominimal case follows along the same lines if $\left[\left(1-\frac{s\|\cdot\|^2}{c^2}\right)_+\right]^{(\frac{1}{2s}-1)}$ is a log-concave function (holds if $s \leq 1/2$). This additional condition implies

$$G_{h,s}^{orlicz}\big(\mathcal{E}_c^s\big) \leq V_h^{(s)}\left(\mathcal{E}_c^s, c^{-n} \cdot \left[\left(1-\frac{s\|\cdot\|^2}{c^2}\right)_+\right]^{(\frac{1}{2s}-1)}\right) = (1+ns) \cdot I_s\big(k_s^c\big) \cdot h(c^{-n}),$$

which provides the necessary upper bound for $G_{h,s}^{orlicz}\big(\mathcal{E}_c^s\big)$. $\square$

Remark. As one would expect, Corollary 5 becomes Corollary 1 if s goes to 0. Note that, when $s > 1/2$, the function $\left[\left(1-\frac{s\|\cdot\|^2}{c^2}\right)_+\right]^{(\frac{1}{2s}-1)}$ is not log-concave (in fact log-convex). Hence, for $h \in \Phi$ and $s > 1/2$, one only has

$$G_{h,s}^{orlicz}\big(\mathcal{E}_c^s\big) \geq (1+ns) \cdot I_s\big(k_s^c\big) \cdot h(c^{-n}).$$

This inequality holds for $h \in \Psi$ and $s > 1/2$ with "$\geq$" replaced by "$\leq$".

3.2 Inequalities

In this subsection, we have additional assumptions for the s-concave function f, that is, f is twice continuous differentiable on S_f, $\det \nabla^2 f \neq 0$ on S_f, $\lim_{x \to \partial S_f} f^s(x) = 0$ and $0 \in S_f$. The collection of all s-concave functions in $\mathcal{C}_s$ with the above addition conditions will be denoted by $\mathcal{C}_s^2$. These assumptions imply that $X_\psi = S_f$ and $X_{\psi_{(s)}^\star} = S_{f_{(s)}^\circ}$. Moreover, as showed in [10], for $f \in \mathcal{C}_s^2$,

$$(1+ns) \cdot I(f) = \int_{X_\psi} (1 - s\psi(x))^{(\frac{1}{s}-1)}\, \widetilde{\psi}(x)\, dx. \tag{24}$$

Consider the function g_1 as follows:

$$g_1(y) = \big(1 - s\psi_{(s)}^\star(y)\big)^{(\frac{1}{s}-1)} \cdot \big(1 + s\langle \nabla \psi_{(s)}^\star(y), y\rangle - s\psi_{(s)}^\star(y)\big). \tag{25}$$

Let $y = T_\psi(x)$. By formula (22), one has,

$$g_1\big(T_\psi(x)\big) = (\widetilde{\psi}(x))^{(1-\frac{1}{s})}(1 - s\psi(x))^{-1}.$$

By formulas (20) and (24), one has, (see also Theorem 4 in [10])

$$I_s(g_1, \psi_{(s)}^\star) = \int_{X_{\psi_{(s)}^\star}} (1 - s\psi_{(s)}^\star(y))^{(\frac{1}{s}-1)} \cdot (1 + s\langle \nabla \psi_{(s)}^\star(y), y\rangle - s\psi_{(s)}^\star(y))\, dy = (1+ns) \cdot I(f_{(s)}^\circ).$$

The following result will be crucial in this subsection. Let g_1 be as in formula (25).

Proposition 3. *Let (f, ψ) with $f \in \mathcal{C}_s^2$ be the pair given by formula (16). Then, for all $h \in \Phi$,*

$$as_{h,s}^{orlicz}(\psi) \leq (1+ns) \cdot I(f) \cdot h\left(\frac{\omega_{n,s}}{I(f_{(s)}^\circ)}\right),$$

and if in addition g_1 is log-concave, then

$$G^{orlicz}_{h,s}(\psi) \leq (1+ns)\cdot I(f)\cdot h\bigg(\frac{\omega_{n,s}}{I(f^\circ_{(s)})}\bigg),$$

Similar inequalities hold for $h \in \Psi$ with " $\leq$ " replaced by " $\geq$ ".

Proof. We only prove the case for $h \in \Phi$ and the proof for $h \in \Psi$ follows along the same line. Let g_1 be as in formula (25). In fact, for $h \in \Phi$,

$$\begin{aligned} as^{orlicz}_{h,s}(\psi) &= \inf\Big\{V^{(s)}_h(\psi, g) : g \in \mathcal{F}^+_{s,\star} \text{ with } I_s(g, \psi^\star_{(s)}) = (1+ns)\cdot\omega_{n,s}\Big\} \\ &\leq \bigg\{V^{(s)}_h\bigg(\psi, \frac{g_1\cdot\omega_{n,s}}{I(f^\circ_{(s)})}\bigg)\bigg\} = (1+ns)\cdot I(f)\cdot h\bigg(\frac{\omega_{n,s}}{I(f^\circ_{(s)})}\bigg). \end{aligned}$$

The results for $G^{orlicz}_{h,s}(\psi)$ follows along the same lines if the additional assumption on g_1 is satisfied. $\square$

Proposition 3 becomes Proposition 1 if $s \to 0$. Moreover, the following cyclic inequalities for $as^{orlicz}_{h,s}(\psi)$ and $G^{orlicz}_{h,s}(\psi)$ hold whose proofs are similar to that for Theorem 4. In fact, Theorem 8 leads to Theorem 4 if $s \to 0$.

Theorem 8. *Let (f, ψ) be the pair given by (16) such that $f \in \mathcal{C}^2_s$. Let g_1 be as in formula (25).*

(i) Assume one of the following conditions: (a) $h \in \Phi$ and $h_1 \in \Psi$ with H increasing; (b) $h, h_1 \in \Phi$ with H decreasing; (c) H concave increasing with either $h, h_1 \in \Phi$ or $h, h_1 \in \Psi$. Then

$$\frac{as^{orlicz}_{h,s}(\psi)}{(1+ns)\cdot I(f)} \;\leq\; H\bigg(\frac{as^{orlicz}_{h_1,s}(\psi)}{(1+ns)\cdot I(f)}\bigg).$$

(ii) Assume one of the following conditions: (d) $h \in \Psi$ and $h_1 \in \Phi$ with H increasing; (e) H convex decreasing with one in Φ and the other one in Ψ; (f) H convex increasing with either $h, h_1 \in \Phi$ or $h, h_1 \in \Psi$. Then

$$\frac{as^{orlicz}_{h,s}(\psi)}{(1+ns)\cdot I(f)} \;\geq\; H\bigg(\frac{as^{orlicz}_{h_1,s}(\psi)}{(1+ns)\cdot I(f)}\bigg).$$

The same inequalities also hold for the Orlicz geominimal surface area, if in addition $g_1 \in \mathcal{L}_{s,\star}$ in conditions (a), (b) and (d).

Let $f \in \mathcal{C}_s$ and $f_z(x) = \big(1 - s\psi(x+z)\big)^{\frac{1}{s}}$ for $z \in \mathbb{R}^n$. Let $(f_z)^\circ_{(s)}$ denote the (s)-Legendre dual of f_z. As proved in [10, 13], there exists $z_0 \in \mathbb{R}^n$ such that

$$I(f_{z_0})\cdot I\big((f_{z_0})^\circ_{(s)}\big) \leq \bigg(\int_{\mathbb{R}^n}\big[\big(1 - s\|x\|^2\big)_+\big]^{\frac{1}{2s}}dx\bigg)^2 = (\omega_{n,s})^2. \tag{26}$$

Equality holds in (26) if and only if there is a positive definite matrix A and a positive constant $\widetilde{c}$, such that $f_{z_0}(x) = \widetilde{c}\cdot\big[\big(1 - s\|Ax\|^2\big)_+\big]^{\frac{1}{2s}}$. For simplicity, let $\mathcal{C}^{2,0}_s$ be the collection of all s-concave functions in $\mathcal{C}^2_s$ such that $z_0 = 0$.

Let c_s and $\bar{c}_s$ be constants defined by

$$c_s = \left(\frac{I\big(f^\circ_{(s)}\big)}{\omega_{n,s}}\right)^{\frac{1}{n}} \quad \text{and} \quad \bar{c}_s = \left(\frac{\omega_{n,s}}{I(f)}\right)^{\frac{1}{n}}.$$

Theorem 9. *Let (f,ψ) be the pair given by formula (16) with $f \in \mathcal{C}_s^{2,0}$.*
(i) Assume that $0 < c_s < \infty$. Then, for $h \in \Phi$,

$$as_{h,s}^{orlicz}(\psi) \;\le\; as_{h,s}^{orlicz}\big(\mathcal{E}^s_{c_s}\big).$$

(ii) Assume that $0 < \bar{c}_s < \infty$. Then, for $h \in \Phi$ be decreasing,

$$as_{h,s}^{orlicz}(\psi) \;\le\; as_{h,s}^{orlicz}(\mathcal{E}^s_{\bar{c}_s}).$$

The above inequality holds for $h \in \Psi$ with "$\le$" replaced by "$\ge$".

There is an equality in (i) and in (ii) if $h \in \Phi$ is strictly decreasing (or $h \in \Psi$ is strictly increasing) if and only if $f(x) = \widetilde{c}\big[\big(1 - s\|Ax\|^2\big)_+\big]^{\frac{1}{2s}}$ for some $\widetilde{c} > 0$ and some positive definite matrix A.

Proof. (i). By inequality (26), one can check that $I(f) \le \omega_{n,s} \cdot c_s^{-n}$. Together with Proposition 3, one has, for all $h \in \Phi$,

$$\begin{aligned} as_{h,s}^{orlicz}(\psi) &\le (1+ns)\cdot I(f)\cdot h\left(\frac{\omega_{n,s}}{I\big(f^\circ_{(s)}\big)}\right) \\ &\le (1+ns)\cdot \omega_{n,s}\cdot c_s^{-n}\cdot h\big(c_s^{-n}\big) \\ &= as_{h,s}^{orlicz}\big(\mathcal{E}^s_{c_s}\big), \end{aligned}$$

where the last equality is due to Corollary 5. Equality holds in the above inequalities only if equality holds in inequality (26). That is, $f(x) = \widetilde{c}\big[\big(1 - s\|Ax\|^2\big)_+\big]^{\frac{1}{2s}}$.

On the other hand, assume that $f(x) = \widetilde{c}\big[\big(1 - s\|Ax\|^2\big)_+\big]^{\frac{1}{2s}}$. Then, equality holds in (26). Identity (24) implies that

$$\frac{\widetilde{c}}{\det A} = \frac{I(f)}{\omega_{n,s}} = \frac{I(f)\cdot I\big(f^\circ_{(s)}\big)}{\omega_{n,s}\cdot I\big(f^\circ_{(s)}\big)} = \frac{\omega_{n,s}}{I\big(f^\circ_{(s)}\big)} = c_s^{-n}.$$

Let $\psi_0 = \frac{1-f^s}{s}$. Then

$$\nabla\psi_0(x) = \frac{\widetilde{c}^s\cdot A^2x}{\big[\big(1 - s\|Ax\|^2\big)_+\big]^{1/2}} \quad \text{and} \quad \widetilde{\psi}_0(x) = \frac{\widetilde{c}^s}{\big[\big(1 - s\|Ax\|^2\big)_+\big]^{1/2}}.$$

Hence, $T_{\psi_0}(x) = A^2x$, and

$$V_h^{(s)}(\psi_0, g) = \int_{\{x:\|Ax\|<s^{-1/2}\}} h\Big(g(A^2x)\cdot\widetilde{c}\cdot\big[\big(1 - s\|Ax\|^2\big)_+\big]^{1-\frac{1}{2s}}\Big)\cdot\widetilde{c}\cdot\big[\big(1 - s\|Ax\|^2\big)_+\big]^{\frac{1}{2s}-1}\,dx.$$

Similar to the proof of Corollary 5, let $g(A^2x) = (\det A)^{-1}\big[\big(1 - s\|Ax\|^2\big)_+\big]^{\frac{1}{2s}-1}$, and then

$$\begin{aligned} as_{h,s}^{orlicz}(\psi_0) &= V_h^{(s)}(\psi_0, g) = \frac{\widetilde{c}}{\det A}\cdot h\Big(\frac{\widetilde{c}}{\det A}\Big)\cdot(1+ns)\cdot\omega_{n,s} \\ &= (1+ns)\cdot\omega_{n,s}\cdot c_s^{-n}\cdot h\big(c_s^{-n}\big) \\ &= as_{h,s}^{orlicz}\big(\mathcal{E}_{c_s}^s\big). \end{aligned}$$

In conclusion, equality holds if and only if $f(x) = \widetilde{c}\big[\big(1 - s\|Ax\|^2\big)_+\big]^{\frac{1}{2s}}$ for some constant $c > 0$ and for some positive definite matrix A.

(ii). By inequality (26), one can check that $I\big(f_{(s)}^\circ\big) \le \omega_{n,s}\cdot(\bar{c})^n$. Together with Proposition 3, one has, for all decreasing $h \in \Phi$

$$\begin{aligned} as_{h,s}^{orlicz}(\psi) &\le (1+ns)\cdot I(f)\cdot h\left(\frac{\omega_{n,s}}{I\big(f_{(s)}^\circ\big)}\right) \\ &\le (1+ns)\cdot\omega_{n,s}\cdot(\bar{c}_s)^{-n}\cdot h\big((\bar{c}_s)^{-n}\big) \\ &= as_{h,s}^{orlicz}(\mathcal{E}_{\bar{c}_s}^s), \end{aligned}$$

where the last equality is due to Corollary 5.

Similar to the characterization of equality in (i), one can prove that if h is strictly decreasing, equality holds in the above inequality if and only if equality holds in inequality (26), that is, $f(x) = \widetilde{c}\big[\big(1 - s\|Ax\|^2\big)_+\big]^{\frac{1}{2s}}$.

Similarly, the desired result for $h \in \Psi$ holds if " $\le$ " is replaced by " $\ge$ ". □

Let g_1 be as in formula (25). If g_1 is a log-concave function and $0 < c_s < \infty$, then for $h \in \Phi$,

$$G_{h,s}^{orlicz}(\psi) \le (1+ns)\cdot\omega_{n,s}\cdot c_s^{-n}\cdot h\big(c_s^{-n}\big).$$

Moreover, If g_1 is a log-concave function and $0 < \bar{c}_s < \infty$, for $h \in \Phi$ being decreasing,

$$G_{h,s}^{orlicz}(\psi) \le (1+ns)\cdot\omega_{n,s}\cdot(\bar{c}_s)^{-n}\cdot h\big((\bar{c}_s)^{-n}\big);$$

while for $h \in \Psi$, the above inequality holds with " $\le$" replaced by " $\ge$". These inequalities together with Corollary 5 and its remark imply the following result.

Corollary 6. *Let (f, ψ) be the pair given by formula (16) with $f \in \mathcal{C}_s^{2,0}$ and let g_1 be log-concave.*

(i) Assume that $0 < c_s < \infty$. Then, for $h \in \Phi$,

$$G_{h,s}^{orlicz}(\psi) \;\le\; G_{h,s}^{orlicz}\big(\mathcal{E}_{c_s}^s\big).$$

(ii) Assume that $0 < \bar{c}_s < \infty$. Then, for $h \in \Phi$ being decreasing,

$$G_{h,s}^{orlicz}(\psi) \;\le\; G_{h,s}^{orlicz}(\mathcal{E}_{\bar{c}_s}^s).$$

The above inequality holds for $h \in \Psi$ with " $\le$" replaced by " $\ge$".

If $s \le 1/2$, there is an equality in (i) and in (ii) if $h \in \Phi$ is strictly decreasing (or $h \in \Psi$ is strictly increasing) if and only if $f(x) = \widetilde{c}\big[\big(1 - s\|Ax\|^2\big)_+\big]^{\frac{1}{2s}}$ for some constant $\widetilde{c} > 0$ and some positive definite matrix A.

Note that Theorem 9 and Corollary 6 would become Corollary 2 if s goes to zero.

3.3 The L_p geominimal surface area of s-concave functions and a Santaló type inequality

The L_p affine surface area of s-concave functions was investigated in [10]. In this subsection, we will briefly discuss the properties for the L_p geominimal surface area of s-concave functions. Taking Theorem 7 into account, it is more natural to define $G_p^{(s)}(\psi)$, the L_p geominimal surface area of the s-concave function f, for $-n \neq p \in \mathbb{R}$, as follows.

Definition 8. *Let (f,ψ) be the pair given by formula (16) with $f \in \mathcal{C}_s$. For $p \geq 0$, define*

$$\begin{aligned} G_p^{(s)}(\psi) &= \left(\frac{1}{1+ns}\right) \cdot \inf_{g \in \mathcal{L}_{s,\star}} \left\{ V_p^{(s)}(\psi,g)^{\frac{n}{n+p}} \; I_s(g,\psi^{\star}_{(s)})^{\frac{p}{n+p}} \right\} \\ &= (\omega_{n,s})^{\frac{p}{n+p}} \cdot \left(\frac{G_{h,s}^{orlicz}(\psi)}{1+ns}\right)^{\frac{n}{n+p}}, \end{aligned}$$

with $h(t)=t^{-p/n}$. For $-n \neq p < 0$, $G_p^{(s)}(\psi)$ is defined similarly but with "$\inf$" replaced by "$\sup$".

The results in previous subsections can be modified accordingly to the L_p geominimal surface area. In particular, it is $SL_{\pm}(n)$-invariant with homogeneous degree $\frac{n(p-n)}{p+n}$. If $c>0$ is a constant and $s \leq 1/2$, Corollary 5 implies that for all $-n \neq p \in \mathbb{R}$,

$$G_p^{(s)}(\mathcal{E}_c^s) = c^{\frac{n(p-n)}{n+p}} \cdot \omega_{n,s}.$$

Moreover, the remark of Corollary 5 implies that if $s > 1/2$, then for $p > 0$,

$$G_p^{(s)}(\mathcal{E}_c^s) \geq c^{\frac{n(p-n)}{n+p}} \cdot \omega_{n,s},$$

and for $-n \neq p < 0$,

$$G_p^{(s)}(\mathcal{E}_c^s) \leq c^{\frac{n(p-n)}{n+p}} \cdot \omega_{n,s}.$$

A direct consequence of Proposition 3 is the following result, which leads to Proposition 2 if $s \to 0$. Similar inequalities were obtained in [10, 11]. Let g_1 be as in formula (25).

Proposition 4. *Let (f,ψ) be the pair given by formula (16) with $f \in \mathcal{C}_s^2$ and $g_1 \in \mathcal{L}_{s,\star}$. Then,*

$$G_p^{(s)}(\psi) \;\leq\; \left[I(f)\right]^{\frac{n}{n+p}} \cdot \left[I(f^{\circ}_{(s)})\right]^{\frac{p}{n+p}},$$

for all $p \geq 0$. Similar inequalities hold for $p \in (-\infty,-n) \cup (-n,0)$ with "$\leq$" replaced by "$\geq$".

Suppose that g_1 and g_2 are log-concave with g_1 as in formula (25) and

$$g_2(y) = \big(1 - s\psi(y)\big)^{(\frac{1}{s}-1)} \cdot \big(1 + s\langle \nabla\psi(y), y\rangle - s\psi(y)\big).$$

Proposition 4 implies that for $p \geq 0$,

$$G_p^{(s)}(\psi) \cdot G_p^{(s)}(\psi^{\star}_{(s)}) \leq I(f) \cdot I(f^{\circ}_{(s)}),$$

while for $-n \neq p < 0$ the above inequality holds with " $\leq$" replaced by " $\geq$". If in addition $f \in \mathcal{C}_s^{2,0}$, the following Santaló type inequality for s-concave functions holds: for $p > 0$,

$$G_p^{(s)}(\psi) \cdot G_p^{(s)}(\psi^{\star}_{(s)}) \leq \omega_{n,s}^2 \leq \left[G_p^{(s)}(\mathcal{E}_1^s)\right]^2.$$

Moreover, if $s \leq 1/2$, there is an equality if and only if $f(x) = \widetilde{c}\left[\left(1 - s\|Ax\|^2\right)_+\right]^{\frac{1}{2s}}$ for some $\widetilde{c} > 0$ and positive definite matrix A.

Immediately from Proposition 4 and inequality (26), one has the following functional L_p affine isoperimetric inequalities for s-concave functions, which becomes Corollary 3 if $s \to 0$. Similar inequalities were obtained in [10, 11]. Let g_1 be as in formula (25).

Corollary 7. *Let (f, ψ) be the pair given by formula (16) with $f \in \mathcal{C}_s^{2,0}$ and $g_1 \in \mathcal{L}_{s,\star}$. Assume that $0 < I(f) < \infty$ and $0 < I(f^{\circ}_{(s)}) < \infty$.*

(i) Let $p > 0$. Then,

$$\frac{G_p^{(s)}(\psi)}{G_p^{(s)}(\mathcal{E}_1^s)} \leq \min\left\{\left(\frac{I(f^{\circ}_{(s)})}{\omega_{n,s}}\right)^{\frac{p-n}{p+n}}, \quad \left(\frac{I(f)}{\omega_{n,s}}\right)^{\frac{n-p}{n+p}}\right\}.$$

(ii) Let $p \in (-n, 0)$. Then,

$$\frac{G_p^{(s)}(\psi)}{G_p^{(s)}(\mathcal{E}_1^s)} \geq \left(\frac{I(f)}{\omega_{n,s}}\right)^{\frac{n-p}{n+p}}.$$

(iii) Let $p < -n$. Then,

$$\frac{G_p^{(s)}(\psi)}{G_p^{(s)}(\mathcal{E}_1^s)} \geq \left(\frac{I(f^{\circ}_{(s)})}{\omega_{n,s}}\right)^{\frac{p-n}{p+n}}.$$

Moreover, if $s \leq 1/2$, there is an equality if and only if $f(x) = \widetilde{c}\left[\left(1 - s\|Ax\|^2\right)_+\right]^{\frac{1}{2s}}$ for some $\widetilde{c} > 0$ and positive definite matrix A.

4 The general mixed Orlicz affine and geominimal surface areas for multiple convex functions

In this section, we introduce the general mixed Orlicz affine and geominimal surface areas for multiple convex functions. We have this notion only for convex functions in this section, but one can introduce it for s-concave functions as well along the same lines.

Let $\vec{h} = (h_1, \cdots, h_m)$, $\vec{g} = (g_1, \cdots, g_m)$, $\vec{F}^1 = (F_1^1, F_2^1, \cdots, F_m^1)$, $\vec{F}^2 = (F_1^2, F_2^2, \cdots, F_m^2)$ etc. We say $\vec{h} \in \Phi^m$ (or $\vec{h} \in \Psi^m$) if each $h_i \in \Phi$ (or $h_i \in \Psi$). Assume that $X_{\vec{\psi}} = \cap_{j=1}^m X_{\psi_j}$ is a nonempty set. Define

$$V_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi}, \vec{g}) = \int_{X_{\vec{\psi}}} \prod_{i=1}^m \left[h_i\left(\frac{g_i(\nabla\psi_i(x))}{F_i^2(\langle x, \nabla\psi_i(x)\rangle - \psi_i(x))}\right) F_i^1(\psi_i(x))\right]^{\frac{1}{m}} dx.$$

If $\psi_i = \psi$, $g_i = g$, $h_i = h$, $F_i^1 = F_1$, and $F_i^2 = F_2$ for all $1 \leq i \leq m$, then $V_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi}, \vec{g})$ becomes $V_{h,F_1,F_2}(\psi, g)$ in Definition 2.

The general mixed Orlicz affine and geominimal surface areas for multiple convex functions are defined as follows. Note that there are many different ways to define mixed Orlicz affine and geominimal surface areas, but we only focus on the one introduced below due to high similarity of their properties.

Definition 9. *Let $F_i^1, F_i^2 : \mathbb{R} \to (0,\infty)$ be measurable functions and $\psi_i \in \mathcal{C}$ for $1 \le i \le m$. For $\vec{h} \in \Phi^m$, the general mixed Orlicz affine surface area of $\vec{\psi}$ is defined by*

$$as^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi}) = \inf\left\{V_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi},\vec{g}) : g_i \in \mathcal{F}^+_{\psi_i^*} \text{ with } I(g_i,\psi_i^*) = (\sqrt{2\pi})^n, \quad 1 \le i \le m\right\},$$

and the general mixed Orlicz geominimal surface area of $\vec{\psi}$ is defined by

$$G^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi}) = \inf\left\{V_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi},\vec{g}) : g_i \in \mathcal{L}_{\psi_i^*} \text{ with } I(g_i,\psi_i^*) = (\sqrt{2\pi})^n, \quad 1 \le i \le m\right\}.$$

The general mixed Orlicz affine and geominimal surface areas of $\vec{\psi}$ for $\vec{h} \in \Psi^m$ are defined similarly with "$\inf$" replaced by "$\sup$".

As before, one can check that $as^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi}) \le G^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi})$ for $\vec{h} \in \Phi^m$ and $as^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi}) \ge G^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi})$ for $\vec{h} \in \Psi^m$. If $F_i^1 = F_i^2 = e^{-t}$ and $\psi_i \in \mathcal{C}$ for $1 \le i \le m$, then $f_i = F_i^1 \circ \psi_i = e^{-\psi_i}$ and $F_i^2 \circ \psi_i^* = e^{-\psi_i^*} = f_i^\circ$ are log-concave functions. Therefore, $as^{orlicz}_{\vec{h}}(\vec{f})$, the mixed Orlicz affine surface area of $\vec{f} = (f_1, \cdots, f_m)$ can be formulated as

$$as^{orlicz}_{\vec{h}}(\vec{f}) = as^{orlicz}_{\vec{h},(e^{-t},\cdots,e^{-t}),(e^{-t},\cdots,e^{-t})}(\vec{\psi}).$$

It is a non-homogeneous extension of the mixed L_p affine surface area of log-concave functions [12]. Similarly, one can define $G^{orlicz}_{\vec{h}}(\vec{f})$, the mixed Orlicz geominimal surface area of $\vec{f}$ by

$$G^{orlicz}_{\vec{h}}(\vec{f}) = G^{orlicz}_{\vec{h},(e^{-t},\cdots,e^{-t}),(e^{-t},\cdots,e^{-t})}(\vec{\psi}).$$

The general mixed Orlicz affine and geominimal surface areas for multiple convex functions are $SL_{\pm}(n)$-invariant. That is, for all $T \in SL_{\pm}(n)$ and $\vec{h} \in \Phi^m \cup \Psi^m$,

$$as^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi} \circ T) = as^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi}), \qquad G^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi} \circ T) = G^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi})$$

where $\vec{\psi} \circ T = (\psi_1 \circ T, \cdots, \psi_m \circ T)$. In particular, $as^{orlicz}_{\vec{h}}(\vec{f})$ and $G^{orlicz}_{\vec{h}}(\vec{f})$ are $SL_{\pm}(n)$-invariant.

A direct consequence of Hölder's inequality is the following Alexander-Fenchel type inequality for the general mixed Orlicz affine and geominimal surface areas for multiple convex functions. Note that the classical Alexandrov-Fenchel inequality for mixed volumes of convex bodies is one of the key inequalities in convex geometry with many applications (see e.g., [30]).

Theorem 10. *Let $\vec{h} \in \Phi^m \cup \Psi^m$ and $F_i^1, F_i^2 : \mathbb{R} \to (0,\infty)$ for all $1 \le i \le m$. Then*

$$\left[as^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi})\right]^m \le \prod_{k=1}^m as^{orlicz}_{h_k,F_k^1,F_k^2}(\psi_k), \qquad \left[G^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi})\right]^m \le \prod_{k=1}^m G^{orlicz}_{h_k,F_k^1,F_k^2}(\psi_k).$$

Moreover, if $\vec{h} \in \Psi^m$, one has, for all $1 \le r \le m-1$,

$$\Big[as^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi})\Big]^r \le \prod_{k=m-r+1}^{m} as^{orlicz}_{\vec{h}_{r,k},\vec{F}^1_{r,k},\vec{F}^2_{r,k}}(\vec{\psi}_{r,k}), \qquad \Big[G^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi})\Big]^r \le \prod_{k=m-r+1}^{m} G^{orlicz}_{\vec{h}_{r,k},\vec{F}^1_{r,k},\vec{F}^2_{r,k}}(\vec{\psi}_{r,k}),$$

where $\vec{F}^1_{r,k}, \vec{F}^2_{r,k}, \vec{h}_{r,k}$ and $\vec{\psi}_{r,k}$ are defined similarly with the following form: for $1 \le r \le m-1$ and $m-r < k \le m$, $\vec{h}_{r,k} = (h_1, h_2, \cdots, h_{m-r}, \underbrace{h_k, \cdots, h_k}_{r})$.

For $1 \le i \le m$, let

$$\hat{c}_i = \left(\frac{I(\breve{F}_i, 1)}{I(F_i^2 \circ \psi_i^*, \psi_i^*)}\right)^{\frac{1}{n}} \text{ and } \bar{c}_i = \left(\frac{I(\breve{F}_i, 1)}{I(F_i^1 \circ \psi_i, \psi_i)}\right)^{\frac{1}{n}},$$

where, for $F_i^1, F_i^2 : \mathbb{R} \to (0,\infty)$, the decreasing function $\breve{F}_i : \mathbb{R} \to (0,\infty)$ is defined by

$$\breve{F}_i(t) = \sup_{\frac{t_1+t_2}{2} \ge t} \sqrt{F_i^1(t_1) F_i^2(t_2)}.$$

The following functional isoperimetric inequality is a direct consequence of Theorems 3 and 10.

Corollary 8. *Let $\psi_i \in \mathcal{C}_0$ and $F_i^1, F_i^2 : \mathbb{R} \to (0,\infty)$ be such that $0 < I(\breve{F}_i, 1) < \infty$ for all $1 \le i \le m$.*
(i) Let $\vec{h} \in \Phi^m$. If $0 < I(F_i^2 \circ \psi_i^*, \psi_i^*) < \infty$ for all $1 \le i \le m$, one has,

$$\Big[as^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi})\Big]^m \ \le \ \prod_{i=1}^{m} as^{orlicz}_{h_i,\breve{F}_i,\breve{F}_i}\Big(\frac{\|\cdot\|^2}{2 \cdot \hat{c}_i^2}\Big),$$

and if in addition $F_i^2 \circ \psi_i^*$ and $\breve{F}_i(\frac{\|\cdot\|^2}{2})$ are log-concave for all $1 \le i \le m$,

$$\Big[G^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi})\Big]^m \ \le \ \prod_{i=1}^{m} G^{orlicz}_{h_i,\breve{F}_i,\breve{F}_i}\Big(\frac{\|\cdot\|^2}{2 \cdot \hat{c}_i^2}\Big).$$

(ii) Let $\vec{h} \in \Phi^m$ with each h_i being decreasing. If $0 < I(F_i^1 \circ \psi_i, \psi_i) < \infty$ for all $1 \le i \le m$, one has

$$\Big[as^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi})\Big]^m \ \le \ \prod_{i=1}^{m} as^{orlicz}_{h_i,\breve{F}_i,\breve{F}_i}\Big(\frac{\bar{c}_i^2 \cdot \|\cdot\|^2}{2}\Big),$$

and if in addition $F_i^2 \circ \psi_i^*$ and $\breve{F}_i(\frac{\|\cdot\|^2}{2})$ are log-concave for all $1 \le i \le m$,

$$\Big[G^{orlicz}_{\vec{h},\vec{F}^1,\vec{F}^2}(\vec{\psi})\Big]^m \ \le \ \prod_{i=1}^{m} G^{orlicz}_{h_i,\breve{F}_i,\breve{F}_i}\Big(\frac{\bar{c}_i^2 \cdot \|\cdot\|^2}{2}\Big).$$

In a similar manner, one can define the i-th general mixed Orlicz affine and geominimal surface areas for two convex functions. Hereafter, let vectors $\vec{h}, \vec{\psi}, \vec{g}, \vec{F}^1$ and $\vec{F}^2$ be as above, but with only 2 coordinates. Assume that $X_{\vec{\psi}} = X_{\psi_1} \cap X_{\psi_2}$ is a nonempty set. Define

$$\begin{aligned} V_{\vec{h},i,\vec{F}^1,\vec{F}^2}(\vec{\psi},\vec{g}) &= \int_{X_{\vec{\psi}}} \left[h_1\left(\frac{g_1(\nabla\psi_1(x))}{F_1^2(\langle x, \nabla\psi_1(x)\rangle - \psi_1(x))} \right) F_1^1(\psi_1(x)) \right]^{\frac{n-i}{n}} \\ &\qquad \times \left[h_2\left(\frac{g_2(\nabla\psi_2(x))}{F_2^2(\langle x, \nabla\psi_2(x)\rangle - \psi_2(x))} \right) F_2^1(\psi_2(x)) \right]^{\frac{i}{n}} dx. \end{aligned}$$

We can define the i-th general mixed Orlicz affine and geominimal surface areas for $\vec{\psi}$ as follows.

Definition 10. *Let $\psi_i \in \mathcal{C}$ and $F_i^1, F_i^2 \colon \mathbb{R} \to (0,\infty)$ be measurable functions for $i = 1, 2$. For $\vec{h} \in \Phi^2$, define the i-th general mixed Orlicz affine surface area for $\vec{\psi}$ by*

$$as^{orlicz}_{\vec{h},i,\vec{F}^1,\vec{F}^2}(\vec{\psi}) = \inf \left\{ V_{\vec{h},i,\vec{F}^1,\vec{F}^2}(\vec{\psi},\vec{g}) : g_i \in \mathcal{F}^+_{\psi_i^*} \text{ with } I(g_1,\psi_1^*) = I(g_2,\psi_2^*) = (\sqrt{2\pi})^n \right\},$$

and define the i-th general mixed Orlicz geominimal surface area for $\vec{\psi}$ by

$$G^{orlicz}_{\vec{h},i,\vec{F}^1,\vec{F}^2}(\vec{\psi}) = \inf \left\{ V_{\vec{h},i,\vec{F}^1,\vec{F}^2}(\vec{\psi},\vec{g}) : g_i \in \mathcal{L}_{\psi_i^*} \text{ with } I(g_1,\psi_1^*) = I(g_2,\psi_2^*) = (\sqrt{2\pi})^n \right\}.$$

For $\vec{h} \in \Psi^2$, $as^{orlicz}_{\vec{h},i,\vec{F}^1,\vec{F}^2}(\vec{\psi})$ and $G^{orlicz}_{\vec{h},i,\vec{F}^1,\vec{F}^2}(\vec{\psi})$ can be defined similarly, with "$\inf$" replaced by "$\sup$".

Let $i < j < k$. For $h_1, h_2 \in \Psi$, Hölder's inequality implies

$$\left[as^{orlicz}_{\vec{h},j,\vec{F}^1,\vec{F}^2}(\vec{\psi}) \right]^{k-i} \leq \left[as^{orlicz}_{\vec{h},i,\vec{F}^1,\vec{F}^2}(\vec{\psi}) \right]^{k-j} \left[as^{orlicz}_{\vec{h},k,\vec{F}^1,\vec{F}^2}(\vec{\psi}) \right]^{j-i}.$$

This inequality (with $i = 0$ and $k = n$) together with Thoerem 3 imply, for instance, the following isoperimetric inequality: for $0 < j < n$,

$$\left[as^{orlicz}_{\vec{h},j,\vec{F}^1,\vec{F}^2}(\vec{\psi}) \right]^n \leq \left[as^{orlicz}_{h_1,\breve{F}_1,\breve{F}_1}\left(\frac{\|\cdot\|^2}{2\cdot \hat{c}_1^2} \right) \right]^{n-j} \left[as^{orlicz}_{h_2,\breve{F}_2,\breve{F}_2}\left(\frac{\|\cdot\|^2}{2\cdot \hat{c}_2^2} \right) \right]^{j},$$

if $\vec{h}, \vec{F}^1, \vec{F}^2$ and $\vec{\psi}$ satisfy the same conditions as those for part (i) in Corollary 8; while if they satisfy the same conditions as those for part (ii) in Corollary 8, then

$$\left[as^{orlicz}_{\vec{h},j,\vec{F}^1,\vec{F}^2}(\vec{\psi}) \right]^n \leq \left[as^{orlicz}_{h_1,\breve{F}_1,\breve{F}_1}\left(\frac{\bar{c}_1^2 \cdot \|\cdot\|^2}{2} \right) \right]^{n-j} \left[as^{orlicz}_{h_2,\breve{F}_2,\breve{F}_2}\left(\frac{\bar{c}_2^2 \cdot \|\cdot\|^2}{2} \right) \right]^{j}.$$

Similar inequalities hold for $G^{orlicz}_{\vec{h},i,\vec{F}^1,\vec{F}^2}(\vec{\psi})$ as long as corresponding conditions verified.

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